Unicity of types for supercuspidals

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1 Introduction

Let F be a non-Archimedean local field with a finite residue field $\mathfrak{k}_F$. Let $\mathfrak{o}_F$ be its complete discrete valuation ring, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, and q_F the size of $\mathfrak{k}_F$. Moreover, let $N > 1$, $G = \mathrm{GL}_N(F)$ and $K = \mathrm{GL}_N(\mathfrak{o}_F)$. Further, let W_F be the Weil group of F and I_F be the inertia group of F. All the representations considered in this paper are over $\mathbb{C}$.

Definition 1.1. *Let π be a smooth irreducible supercuspidal representation of G, then we define the **inertial support** $\mathfrak{I}(\pi)$ of π to be:*

$$\mathfrak{I}(\pi) = \{\pi' : \pi' \cong \pi \otimes \chi \circ \det\}$$

where χ is some unramified quasicharacter of $F^\times$.

Definition 1.2. *Suppose H is a compact open subgroup of G, τ a smooth irreducible representation of H and π a smooth irreducible supercuspidal representation of G, then (H, τ) is a **type** for $\mathfrak{I}(\pi)$, if for all smooth irreducible representations π' of G:*

$$\pi' \left|_H \right. \textit{ contains } \tau \Leftrightarrow \pi' \in \mathfrak{I}(\pi)$$

where χ is some unramified quasicharacter of $F^\times$.

Our main result is:

Theorem 1.3. *Let π be a smooth irreducible supercuspidal representation of G, then there exists a smooth irreducible representation τ of K depending on $\mathfrak{I}(\pi)$, such that (K, τ) is a type for $\mathfrak{I}(\pi)$. Moreover, τ is unique (up to isomorphism) and it occurs in $\pi \left|_K \right.$ with multiplicity one.*

This implies a kind of inertial local Langlands correspondence:

Corollary 1.4. *Let φ be a smooth N-dimensional representation of I_F, which extends to a smooth <u>irreducible</u> Frobenius semisimple representation of W_F, then there exists a unique (up to isomorphism) smooth irreducible representation $\tau(\varphi)$ of K, such that for any smooth irreducible infinite dimensional representation π of G:*

$$\pi \left|_K \right. \textit{ contains } \tau(\varphi) \Leftrightarrow \mathrm{WD}(\pi) \left|_{I_F} \right. \cong \varphi$$

where $\mathrm{WD}(\pi)$ is a Weil-Deligne representation of W_F corresponding to π via the local Langlands correspondence.

Our result and methods generalise the case, when $N = 2$, which was considered by G. Henniart in [9]. The paper heavily relies on the classification of supercuspidals due to C. Bushnell and P. Kutzko. The existence of such τ is almost immediate from [5](6.2.3), the difficult part is proving uniqueness.

The paper is structured as follows. We recall some facts and definitions from Bushnell-Kutzko theory in sections 2.2-2.4. In section 2.6 we introduce some of our own notation. From Bushnell-Kutzko theory we know that every supercuspidal representation is induced from an open compact-mod-centre subgroup of G. The restriction of π to K results in a Mackey's decomposition. The representation coming from the double coset, which contains 1, is our τ. We prove that in section 3. Then in section 4 we prove that under certain conditions an irreducible summand of $\pi\,|_K$ cannot be a type. In section 5 we choose a nice representative from each double coset. Then we have to consider two different cases, namely sections 6 and 7. The idea is that unless the double coset contains 1, then any irreducible summand of the representation coming from the double coset in the Mackey's decomposition of $\pi\,|_K$ cannot be a type by section 4. Finally in section 8 we prove the main result.

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2 Notation and Preliminaries

Let F be a non-Archimedean local field with a finite residue field $\mathfrak{k}_F$. Let $\mathfrak{o}_F$ be its complete discrete valuation ring, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, and q_F the size of $\mathfrak{k}_F$. Moreover, let V be an F-vector space of finite dimension $N > 1$, $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Further, let ψ_F be a fixed continuous additive character of the group F, with conductor $\mathfrak{p}_F$, and let

$$\psi_A = \psi_F \circ \mathrm{tr}_{A/F}\,.$$

2.1 Intertwining

If H is a subgroup of G and ρ, τ are representations of H, let

$$\langle \rho, \tau \rangle_H = \dim_{\mathbb{C}} \mathrm{Hom}_H(\rho, \tau).$$

If $g \in G$, then let

$$H^g = gHg^{-1}$$

and ρ^g, be a representation of H^g,

$$\rho^g(x) = \rho(g^{-1}xg), \, \forall x \in H^g.$$

We say g *intertwines* τ and ρ in G, if

$$\langle \tau, \rho^g \rangle_{H \cap H^g} \neq 0.$$

The set of all $g \in G$, which intertwine τ and ρ is called the *intertwining* of τ and ρ and is denoted by $I_G(\tau, \rho|H)$.

2.2 Hereditary orders

For a complete account of hereditary orders we refer the reader to [5]§1, [1] and [3]. Everything below is taken from [5]§1.1.

Let $\mathfrak{A}$ be an $\mathfrak{o}_F$-order in A, then $\mathfrak{A}$ is (left) *hereditary* if every (left) $\mathfrak{A}$-lattice is $\mathfrak{A}$-projective.

2.2.1 Lattice chains

An $\mathfrak{o}_F$-*lattice chain* $\mathcal{L}$ in V is a sequence $\{L_i : i \in \mathbb{Z}\}$, such that

(i) $L_{i+1} \subsetneq L_i$, $i \in \mathbb{Z}$

(ii) there exists $e \in \mathbb{Z}$ such that $\pi_F L_i = L_{i+e}$ for all $i \in \mathbb{Z}$.

The integer $e = e(\mathcal{L})$ is uniquely determined and is called an $\mathfrak{o}_F$-*period* of $\mathcal{L}$.

2.2.2 Hereditary orders

Hereditary orders in A are in bijection with lattice chains in V. Given a lattice chain $\mathcal{L}$ we define:

$$\mathrm{End}^n_{\mathfrak{o}_F}(\mathcal{L}) = \{x \in A : xL_i \subseteq L_{i+n}, i \in \mathbb{Z}\}$$

for each $n \in \mathbb{Z}$. Then $\mathfrak{A} = \mathfrak{A}(\mathcal{L}) = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L})$ is a hereditary $\mathfrak{o}_F$-order in A. We can recover $\mathcal{L}$ from $\mathfrak{A}$ up to a shift in the index: $\mathcal{L}$ is the set of all $\mathfrak{o}_F$-lattices in V, which are $\mathfrak{A}$ modules. The lattices $\mathrm{End}^n_{\mathfrak{o}_F}(\mathcal{L})$ are $(\mathfrak{A}, \mathfrak{A})$-bimodules. Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$, then

$$\mathfrak{P} = \mathrm{End}^1_{\mathfrak{o}_F}(\mathcal{L}).$$

As a fractional ideal of $\mathfrak{A}$, the radical $\mathfrak{P}$ is invertible, and we have:

$$\mathfrak{P}^n = \mathrm{End}^n_{\mathfrak{o}_F}(\mathcal{L}),\ n \in \mathbb{Z}.$$

In particular,

$$\mathfrak{P}^n L_i = L_{i+n},\ i, n \in \mathbb{Z}.$$

The $\mathfrak{o}_F$-period e of $\mathcal{L}$ is also a function of $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, given by

$$\mathfrak{p}_F \mathfrak{A} = \mathfrak{P}^e.$$

So we will write $e = e(\mathcal{L}) = e(\mathfrak{A}|\mathfrak{o}_F)$. We define a sequence of compact open subgroups in G:

$$\mathbf{U}^0(\mathfrak{A}) = \mathbf{U}(\mathfrak{A}) = \mathfrak{A}^\times,$$

$$\mathbf{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}^n,\ n \geq 1.$$

Also, for $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, we set

$$\mathfrak{K}(\mathfrak{A}) = \{x \in G : xL_i \in \mathcal{L}, i \in \mathbb{Z}\} = \{x \in G : x^{-1}\mathfrak{A}x = \mathfrak{A}\}.$$

This is an open compact-mod-centre subgroup of G, and the $\mathbf{U}^n(\mathfrak{A})$, for $n \geq 0$, are normal subgroups of it. In particular, $\mathbf{U}(\mathfrak{A})$ is the unique maximal compact open subgroup of $\mathfrak{K}(\mathfrak{A})$.

There is also a "valuation" map associated with the hereditary order $\mathfrak{A}$. Define:

$$\nu_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} : x \in \mathfrak{P}^n\},\ x \in A,$$

with the understanding that $\nu_{\mathfrak{A}}(0) = \infty$. In particular, if $x \in \mathfrak{K}(\mathfrak{A})$, then $\nu_{\mathfrak{A}}(x) = n$, where

$$x\mathfrak{A} = \mathfrak{A}x = \mathfrak{P}^n.$$

This induces an exact sequence:

$$1 \longrightarrow \mathbf{U}(\mathfrak{A}) \longrightarrow \mathfrak{K}(\mathfrak{A}) \xrightarrow{\nu_{\mathfrak{A}}} \mathbb{Z}$$

Given $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ we have a canonical isomorphism

$$\mathfrak{A}/\mathfrak{P} \cong \prod_{i=0}^{e-1} \mathrm{End}_{\mathfrak{k}_F}(L_i/L_{i+1}),$$

where $e = e(\mathfrak{A}|\mathfrak{o}_F)$. And, we can always choose a basis for V, such that $\mathfrak{A}$ is identified with block upper triangular matrices modulo $\mathfrak{p}_F$. We say $\mathfrak{A}$ is *principal* if $(L_i : L_{i+1}) = (L_0 : L_1)$, for all i. In that case, every block has size $\frac{N}{e} \times \frac{N}{e}$.

2.2.3 The character ψ_b

Let n and m be integers satisfying $n > m \geq [\frac{n}{2}] \geq 0$, where $[x]$ denotes the greatest integer $\leq x$, for $x \in \mathbb{R}$. We then have a canonical isomorphism

$$\mathbf{U}^{m+1}(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A}) \xrightarrow{\cong} \mathfrak{P}^{m+1}/\mathfrak{P}^{n+1},$$

given by $x \mapsto x - 1$. This leads to an isomorphism

$$(\mathbf{U}^{m+1}(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A}))^{\wedge} \xrightarrow{\cong} \mathfrak{P}^{-n}/\mathfrak{P}^{-m},$$

where "hat" $^\wedge$ denotes Pontryagin dual. Explicitly, this is given by

$$b + \mathfrak{P}^{-m} \mapsto \psi_{A,b} = \psi_b,\ b \in \mathfrak{P}^{-n}, \text{ where}$$

$$\psi_b(1 + x) = \psi_A(bx),\ x \in \mathfrak{P}^{m+1}.$$

2.3 Strata

For details we refer the reader to [5]§1.

A *stratum* is a 4-tuple $[\mathfrak{A}, n, m, b]$ consisting of a hereditary order $\mathfrak{A}$, integers $n > m$, and $b \in A$, such that $\nu_{\mathfrak{A}}(b) \geq -n$.

2.3.1 Equivalence

We define an equivalence relation on the set of strata:

$$[\mathfrak{A}_1, n_1, m_1, b_1] \sim [\mathfrak{A}_2, n_2, m_2, b_2], \text{ if}$$

$$b_1 + \mathfrak{P}_1^{-m_1} = b_2 + \mathfrak{P}_2^{-m_2}.$$

Equivalence implies $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$, $m_1 = m_2$. Moreover, if $n_1 = -\nu_{\mathfrak{A}}(b_1)$ and $n_2 = -\nu_{\mathfrak{A}}(b_2)$, then $n_1 = n_2$, see [5](1.5.2).

2.3.2 Simple strata

A stratum $[\mathfrak{A}, n, m, \beta]$ is *pure* if

(i) the algebra $E = F[\beta]$ is a field,

(ii) $E^\times \subset \mathfrak{K}(\mathfrak{A})$,

(iii) $\nu_{\mathfrak{A}}(\beta) = -n$.

It is called *simple* if, in addition

(iv) $m < -k_0(\beta, \mathfrak{A})$.

The definition of $k_0(\beta, \mathfrak{A})$ is rather technical, so we refer the reader to [5](1.4.5). We will need to know that $k_0(\beta, \mathfrak{A})$ is an integer and

$$k_0(\beta, \mathfrak{A}) \geq \nu_{\mathfrak{A}}(\beta).$$

Suppose $[\mathfrak{A}, n, m, \beta]$ is a simple stratum, then we define

$$B_\beta = \{x \in A : \beta x = x\beta\} = \mathrm{End}_E(V).$$

Let

$$\mathfrak{B}_\beta = \mathfrak{A} \cap B_\beta.$$

Since $E^\times \subset \mathfrak{K}(\mathfrak{A})$, we can view $\mathcal{L}$ as an $\mathfrak{o}_E$ lattice chain. Hence $\mathfrak{B}_\beta$ is a hereditary order in B_β. We define

$$\mathfrak{Q}_\beta^n = \mathfrak{P}^n \cap B_\beta,$$

$$\mathbf{U}(\mathfrak{B}_\beta) = \mathbf{U}(\mathfrak{A}) \cap B_\beta$$

and

$$\mathbf{U}^n(\mathfrak{B}_\beta) = \mathbf{U}^n(\mathfrak{A}) \cap B_\beta.$$

All the notions above coincide with the ones defined for $\mathfrak{A}$. We also have

$$e(\mathfrak{B}_\beta|\mathfrak{o}_E)e(E|F) = e(\mathfrak{A}|\mathfrak{o}_F),$$

where $e(E|F)$ is a ramification index of E over F, since $\pi_E^{e(E|F)}.L_i = \pi_F.L_i$, for all $L_i \in \mathcal{L}$.

2.3.3 Tame corestriction

Let E/F be a subfield of A, with the centraliser B. A *tame corestriction* on A relative to E/F, is a (B, B)-bimodule homomorphism, $s : A \to B$, such that $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for every hereditary $\mathfrak{o}_F$ order $\mathfrak{A}$ in A, which is normalised by $E^\times$. We will need the following result [5](1.3.4).

Let ψ_E be a continuous additive character of E, with conductor $\mathfrak{p}_E$ and let $\psi_B = \psi_E \circ \mathrm{tr}_{B/E}$, then there exists a unique map $s : A \to B$, such that

$$\psi_A(ab) = \psi_B(s(a)b),\ \forall a \in A,\ \forall b \in B.$$

This map is a tame corestriction relative to E/F.

2.3.4 Approximation of simple strata

We will use the following result [5](2.4.1).

(i) Let $[\mathfrak{A}, n, m, \beta]$ be a pure stratum in A, then there exists a simple stratum $[\mathfrak{A}, n, m, \gamma]$ in A, such that

$$[\mathfrak{A}, n, m, \beta] \sim [\mathfrak{A}, n, m, \gamma].$$

Among all the pure stratum $[\mathfrak{A}, n, m, \beta']$ equivalent to $[\mathfrak{A}, n, m, \beta]$ the simple ones are precisely those for which the field extension $F[\beta']/F$ has minimal degree.

(ii) Let $[\mathfrak{A}, n, m, \gamma_1]$, $[\mathfrak{A}, n, m, \gamma_2]$ be simple strata in A, which are equivalent to each other, then

$$k_0(\gamma_1, \mathfrak{A}) = k_0(\gamma_2, \mathfrak{A}).$$

(iii) Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A, with $r = -k_0(\beta, \mathfrak{A})$. Let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum in A which is equivalent to $[\mathfrak{A}, n, r, \beta]$, let s_γ be a tame corestriction on A relative to $F[\gamma]/F$, let B_γ be the A-centraliser of γ, and $\mathfrak{B}_\gamma = \mathfrak{A} \cap B_\gamma$. Then $[\mathfrak{B}_\gamma, r, r-1, s_\gamma(\beta - \gamma)]$ is equivalent to a simple stratum in B_γ.

We will also need [5](2.2.8).

Let $[\mathfrak{A}, n, m, \beta]$ be a simple stratum in A. Let B be the A-centraliser of $E = F[\beta]$, and $\mathfrak{B} = B \cap \mathfrak{A}$. Let $b \in A$ with $\nu_{\mathfrak{A}}(\beta) = -r$, and let s be a tame corestriction on A relative to $F[\beta]/F$. Suppose that the stratum $[\mathfrak{B}, m, m-1, s(b)]$ is equivalent to some simple stratum $[\mathfrak{B}, m, m-1, c]$ in B. Then $[\mathfrak{A}, n, m-1, \beta + b]$ is equivalent to a simple stratum $[\mathfrak{A}, n, m-1, \beta_1]$. Moreover, if $E_1 = F[\beta, c]$, $K = F[\beta_1]$, we have

(i) $e(K|F) = e(E_1|F)$, $f(K|F) = f(E_1|F)$;

(ii) $k_0(\beta_1, \mathfrak{A}) = \max\{k_0(\beta, \mathfrak{A}),\ k_0(c, \mathfrak{B})\}$.

2.4 Simple types

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and let $r = -k_0(\beta, \mathfrak{A})$.

2.4.1 Groups $J(\beta, \mathfrak{A})$ and $H(\beta, \mathfrak{A})$

To a simple stratum $[\mathfrak{A}, n, 0, \beta]$ we can associate compact open subgroups of $\mathbf{U}(\mathfrak{A})$: $J(\beta, \mathfrak{A})$ and $H(\beta, \mathfrak{A})$, see [5](3.1.14). Both of them have natural filtrations by normal subgroups:

$$J^m(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap \mathbf{U}^m(\mathfrak{A}),$$

$$H^m(\beta, \mathfrak{A}) = H(\beta, \mathfrak{A}) \cap \mathbf{U}^m(\mathfrak{A}).$$

The groups $J^m(\beta, \mathfrak{A})$ and $H^m(\beta, \mathfrak{A})$ are normalised by $\mathfrak{K}(\mathfrak{B}_\beta)$, for all $m \geq 0$. Moreover, $H(\beta, \mathfrak{A})$ is a subgroup of $J(\beta, \mathfrak{A})$ and $H^m(\beta, \mathfrak{A})$ are normal in $J(\beta, \mathfrak{A})$, for $m \geq 1$. We will drop various indices, when the meaning is clear.

We have the following decompositions: for $0 \leq m \leq [\frac{r}{2}] + 1$,

$$H^m(\beta, \mathfrak{A}) = \mathbf{U}^m(\mathfrak{B}_\beta) H^{[\frac{r}{2}]+1}(\beta, \mathfrak{A})$$

and for $0 \leq m \leq [\frac{r+1}{2}]$

$$J^m(\beta, \mathfrak{A}) = \mathbf{U}^m(\mathfrak{B}_\beta) J^{[\frac{r+1}{2}]}(\beta, \mathfrak{A})$$

where square brackets denote the integer part, see [5](3.1.15).

2.4.2 Simple characters $\mathcal{C}(\mathfrak{A}, m, \beta)$

We can define a very special set of linear characters $\mathcal{C}(\mathfrak{A}, m, \beta)$ of $H^{m+1}(\beta, \mathfrak{A})$, called *simple characters*, see [5](3.2). We will need the following properties:

For $0 \leq m \leq [\frac{r}{2}]$ the restriction induces a surjective map

$$\mathcal{C}(\mathfrak{A}, m, \beta) \to \mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \beta)$$

The fibres of this map are of the form $\theta.X$, where $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ and X is the group of characters of $\mathbf{U}^{m+1}(\mathfrak{B}_\beta)/\mathbf{U}^{[\frac{r}{2}]}(\mathfrak{B}_\beta)$, which factor through the determinant $\det_{B_\beta}$, see [5](3.2.5).

If $n = 1$, then $H^1(\beta, \mathfrak{A}) = J^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{A})$ and $\mathcal{C}(\mathfrak{A}, 0, \beta) = \{\psi_\beta\}$, see [5](3.1.7) and (3.2.1).

2.4.3 Intertwining of simple characters

For $0 \leq m \leq [\frac{r}{2}]$ and $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$, we have

$$I_G(\theta, \theta | H^{m+1}(\beta, \mathfrak{A})) = J^1(\beta, \mathfrak{A}) B_\beta^\times J^1(\beta, \mathfrak{A})$$

see [5](3.3.2).

For $i = 1, 2$, let $[\mathfrak{A}, n, m, \beta_i]$ be simple strata with $m \geq 0$. Suppose there exists $\theta_i \in \mathcal{C}(\mathfrak{A}, m, \beta_i)$, which intertwine in G. Then there exists $x \in \mathbf{U}(\mathfrak{A})$ such that

$$\mathcal{C}(\mathfrak{A}, m, \beta_1) = \mathcal{C}(\mathfrak{A}, m, x^{-1}\beta_2 x)$$

and conjugation by x carries θ_1 to θ_2, see [5](3.5.11).

If $n > 1$ and $r = 1$ and $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, then there exists a simple stratum $[\mathfrak{A}, n, 1, \gamma]$, such that $[\mathfrak{A}, n, 1, \beta] \sim [\mathfrak{A}, n, 1, \gamma]$, $H^1(\beta, \mathfrak{A}) = H^1(\gamma, \mathfrak{A})$ and

$$\theta = \theta_0 \psi_c$$

where $\theta_0 \in \mathcal{C}(\mathfrak{A}, 0, \gamma)$ and $c = \beta - \gamma$, see [5](3.2.3).

Moreover, $I_G(\theta, \theta_0 | H^1(\gamma, \mathfrak{A})) = \emptyset$, see [5](3.5.12).

2.4.4 Representations η and κ

If $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, then there exists a unique irreducible smooth representation η of $J^1(\beta, \mathfrak{A})$, such that $\eta|_{H^1}$ contains θ, see [5](5.1.1).

Given η, there exists a smooth irreducible representation κ of $J(\beta, \mathfrak{A})$, such that $\kappa|_{J^1} \cong \eta$ and $B_\beta^\times \subset I_G(\kappa, \kappa | J)$. We say κ is a *β-extension* of η, see [5](5.2.2).

2.4.5 Simple types

A *simple type* in G is one of the following [5](5.5.10):

1. An irreducible representation $\lambda = \kappa \otimes \sigma$ of $J = J(\beta, \mathfrak{A})$, where $\mathfrak{A}$ is a principal $\mathfrak{o}_F$ order in A, $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum. For some $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, κ is a β-extension of η, the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ containing θ. Let $E = F[\beta]$, then

$$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta) \cong \mathrm{GL}_f(\mathfrak{k}_E)^e$$

for some integers e and f and σ is a lift of representation $\sigma_0 \otimes \ldots \otimes \sigma_0$, where σ_0 is an irreducible cuspidal representation of $GL_f(\mathfrak{k}_E)$.

2. An irreducible representation σ of $\mathbf{U}(\mathfrak{A})$, where $\mathfrak{A}$ is a principal $\mathfrak{o}_F$ order in A. We have $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A}) \cong \mathrm{GL}_f(\mathfrak{k}_F)^e$, for some integers e and f. Then σ is a lift of $\sigma_0 \otimes \ldots \otimes \sigma_0$, where σ_0 is an irreducible cuspidal representation of $GL_f(\mathfrak{k}_F)$.

The second part, can be viewed as a special case of the first part, with trivial character as a simple character, $F = F[\beta]$ and $\mathfrak{B}_\beta = \mathfrak{A}$.

2.5 Supercuspidal representations

Let (J, λ) be a simple type, with the simple stratum $[\mathfrak{A}, n, 0, \beta]$ and $E = F[\beta]$.

2.5.1 Maximal simple types

The following are equivalent:

(i) $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$

(ii) There exists an irreducible supercuspidal representation π of G, such that $\pi|_J$ contains λ

(iii) Any irreducible representation π of G, such that $\pi|_J$ contains λ, is supercuspidal.

Suppose these conditions hold, and let π be an irreducible representation of G, which contains λ. Then an irreducible representation π' will contain λ if and only if $\pi' \cong \pi \otimes \chi \circ \det$, for some unramified quasicharacter χ of $F^\times$. We say that such (J, λ) is a *maximal* simple type, see [5](6.2.3).

We also note, that if (J, λ) is a maximal simple type then

$$I_G(\lambda, \lambda|J) = E^\times J,$$

see [5](5.5.11) and (6.2.1).

2.5.2 Structure of supercuspidal representations

Let π be an irreducible supercuspidal representation of G. There exists a simple type (J, λ) in G, such that $\pi|_J$ contains λ. Further,

(i) the simple type (J, λ) is uniquely determined up to G-conjugacy.

(ii) if (J, λ) is given by a simple stratum $[\mathfrak{A}, n, 0, \beta]$ with $E = F[\beta]$, there is a uniquely determined representation Λ of $E^\times J$, such that $\Lambda|_J \cong \lambda$ and $\pi \cong \text{c-Ind}_{E^\times J}^G \Lambda$.

(iii) If $(J, \lambda) = (\mathbf{U}(\mathfrak{A}), \sigma)$, i.e., $J = \mathbf{U}(\mathfrak{A})$ for some principal $\mathfrak{o}_F$ order $\mathfrak{A}$ and λ is trivial on $\mathbf{U}^1(\mathfrak{A})$, then there exists a uniquely determined representation Λ of $F^\times \mathbf{U}(\mathfrak{A})$, such that $\Lambda|_{F^\times \mathbf{U}(\mathfrak{A})} \cong \lambda$ and $\pi \cong \text{c-Ind}_{F^\times \mathbf{U}(\mathfrak{A})}^G \Lambda$.

See [5](8.4.1). Here c-Ind denotes compact induction, which is described in detail in [2].

2.5.3 Split types

A split type is a pair (K', ϑ), where K' is a compact open subgroup of G, and ϑ is an irreducible representation K'. There are four flavours of split types, and we will define the three, that we require in the course of the paper. We will need to use the following result:

Let π' be a smooth irreducible representation of G. If $\pi'|_{K'}$ contains ϑ, then the Jacquet module π'_U is nontrivial for some unipotent radical of a proper parabolic subgroup of G, see [5](8.2.5) and (8.3.3).

In particular, a supercuspidal representation cannot contain a split type.

2.6 The Setup

In this paper every hereditary $\mathfrak{o}_F$ order $\mathfrak{A}$ in A will come with a lattice chain:

$$\mathcal{L} : \ldots L_{i+1} \subset L_i \subset L_{i-1} \ldots$$

such that $\mathfrak{A} = \text{End}_{\mathfrak{o}_F}(\mathcal{L})$. The lattice chain $\mathcal{L}$ will come with a basis $v_1, \ldots, v_N$ of V, with respect to which $\mathfrak{A}$ is identified with the ring of block upper triangular matrices modulo $\mathfrak{p}_F$. If $\mathfrak{A}$ is principal, then:

$$L_0 = \mathfrak{o}_F v_1 + \ldots + \mathfrak{o}_F v_N$$

$$L_i = \mathfrak{o}_F v_1 + \ldots + \mathfrak{o}_F v_{\frac{N}{e}(e-i)} + \mathfrak{p}_F v_{\frac{N}{e}(e-i)+1} + \ldots + \mathfrak{p}_F v_N$$

for $0 < i < e = e(\mathfrak{A}|\mathfrak{o}_F)$. We define a useful element Π on the basis of V.

$$\Pi : v_i \mapsto \pi_F v_{\frac{N}{e}(e-1)+i}, \text{ for } 1 \leq i \leq \frac{N}{e}$$
$$v_j \mapsto v_{j-\frac{N}{e}}, \text{ for } \frac{N}{e} + 1 \leq j \leq N.$$

By inspecting how Π acts on the lattice chain $\mathcal{L}$, we see that $\Pi \in \mathfrak{K}(\mathfrak{A})$ and $\nu_{\mathfrak{A}}(\Pi) = 1$. Hence we have a short exact sequence:

$$1 \longrightarrow \mathbf{U}(\mathfrak{A}) \longrightarrow \mathfrak{K}(\mathfrak{A}) \xrightarrow{\nu_{\mathfrak{A}}} \mathbb{Z} \longrightarrow 0$$

We will always denote

$$K = \mathrm{Aut}_{\mathfrak{o}_F}(L_0)$$

So $\mathbf{U}(\mathfrak{A})$ is always a subgroup of K and with respect to our basis K is identified with $\mathrm{GL}_N(\mathfrak{o}_F)$.

Throughout the paper we fix a supercuspidal representation π of G. Let (J, λ) be a simple type occurring in π, with a simple stratum $[\mathfrak{A}, n, 0, \beta]$, $E = F[\beta]$. We define

$$\rho = \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$$

Since $I_G(\lambda, \lambda | J) = E^\times J$, and $E^\times J \cap \mathbf{U}(\mathfrak{A}) = J$, as J is the unique maximal compact open subgroup of $E^\times J$, the representation ρ is irreducible. It is worth writing out the details, since we will use this kind of argument a lot:

$$\langle \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda, \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda \rangle_{\mathbf{U}(\mathfrak{A})} = \langle \lambda, \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda \rangle_J = \dots$$

$$\dots = \sum_{u \in J \backslash \mathbf{U}(\mathfrak{A}) / J} \langle \lambda, \mathrm{Ind}_{J \cap J^u}^J \lambda^u \rangle_J = \sum_{u \in J \backslash \mathbf{U}(\mathfrak{A}) / J} \langle \lambda, \lambda^u \rangle_{J \cap J^u}$$

the equalities above involve Frobenius reciprocity and Mackey's formula, hence $\langle \rho, \rho \rangle_{\mathbf{U}(\mathfrak{A})} = 1$. Since $[\mathfrak{A}, n, 0, \beta]$ is simple, $\mathfrak{A}$ is principal and since (J, λ) is contained in a supercuspidal representation, we have $e(\mathfrak{B}_\beta | \mathfrak{o}_E) = 1$, hence $\nu_{\mathfrak{A}}(\pi_E) = 1$. That implies

$$\mathfrak{K}(\mathfrak{A}) = E^\times \mathbf{U}(\mathfrak{A})$$

Let Λ be the unique extension of λ to $E^\times J$ such that $\pi \cong \text{c-Ind}_{E^\times J}^G \Lambda$. We define

$$\tilde{\rho} = \mathrm{Ind}_{E^\times J}^{\mathfrak{K}(\mathfrak{A})} \Lambda$$

Just by transitivity of induction, we have

$$\pi \cong \text{c-Ind}_{\mathfrak{K}(\mathfrak{A})}^G \tilde{\rho}$$

Since π is irreducible, $\tilde{\rho}$ is also irreducible and

$$\tilde{\rho}\,|_{\mathbf{U}(\mathfrak{A})} \cong \rho$$

since in this case there is only one double coset in Mackey's formula.

Now we forget all about our original (J, λ). There are two justifications for this. The vague is: since $\pi \cong \text{c-Ind}_{\mathfrak{K}(\mathfrak{A})}^{G} \tilde{\rho}$, we do not lose any information. The rigorous one is: suppose (J_1, λ_1) is another simple type contained in π, with a simple stratum $[\mathfrak{A}, n_1, 0, \beta_1]$, then there exists $x \in \mathbf{U}(\mathfrak{A})$, such that $(J, \lambda) = (J_1^x, \lambda_1^x)$, and hence $\rho \cong \text{Ind}_{J_1}^{\mathbf{U}(\mathfrak{A})} \lambda_1$ and $\tilde{\rho} \cong \text{Ind}_{E_1^\times J_1}^{\mathfrak{K}(\mathfrak{A})} \Lambda_1$. To see that, one needs to go through the proof of [5](5.7.1), which says "intertwining implies conjugacy", and in the last step use that $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$. As this does not affect us, we will not provide the details.

Why do we prefer working with ρ, rather than with a simple type? We are interested in the irreducible summands of $\pi|_K$, and

$$\pi|_K \cong \bigoplus_{g \in K \backslash G / \mathfrak{K}(\mathfrak{A})} \text{Ind}_{K \cap \mathfrak{K}(\mathfrak{A})^g}^{K} \tilde{\rho}^g|_{K \cap \mathfrak{K}(\mathfrak{A})^g}$$

Since, $\mathbf{U}(\mathfrak{A})$ is the unique maximal compact open subgroup of $\mathfrak{K}(\mathfrak{A})$

$$\pi|_K \cong \bigoplus_{g \in K \backslash G / \mathfrak{K}(\mathfrak{A})} \text{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^{K} \rho^g|_{K \cap \mathbf{U}(\mathfrak{A})^g}$$

In order to acquire some information about the irreducible summands, we will need to choose some nice representative of a double coset $Kg\mathfrak{K}(\mathfrak{A})$. Now $\mathfrak{K}(\mathfrak{A})$ is reasonable to work with, since we can identify $\mathfrak{A}$ with block upper triangular matrices modulo $\mathfrak{p}_F$, and $\nu_{\mathfrak{A}}(\Pi) = 1$, so Π and $\mathbf{U}(\mathfrak{A})$ will generate $\mathfrak{K}(\mathfrak{A})$. On the other hand, it would be a lot harder to work with double cosets $KgE^\times J$, just ask: for what matrices β is $[\mathfrak{A}, n, 0, \beta]$ a simple stratum, if $\mathfrak{A}$ is identified with the block upper triangular matrices modulo $\mathfrak{p}_F$?

Since, by the argument above, ρ determines the restriction of π to any compact open subgroup of G, we will often omit π from the statements of propositions, and will work with ρ instead.

3 Existence

Proposition 3.1. *Let $\tau = \text{Ind}_{\mathbf{U}(\mathfrak{A})}^{K} \rho$, then τ is a type for $\mathfrak{I}(\pi)$. Moreover, τ occurs in $\pi|_K$ with multiplicity one.*

Proof. Let (J, λ) be a simple type, with a simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that $\rho \cong \text{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$. Let $E = F[\beta]$. By [5](5.5.11) coupled with [5](6.2.1) we know

$$I_G(\lambda, \lambda|J) = E^\times J, \text{ and } E^\times J \cap K = J$$

since J is the unique maximal open compact subgroup of $E^\times J$, hence $\tau \cong \operatorname{Ind}_J^K \lambda$ is irreducible.

$$\pi\,|_K \cong \bigoplus_{g \in K\backslash G/\mathfrak{K}(\mathfrak{A})} \operatorname{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^K \rho^g\,|_{K \cap \mathbf{U}(\mathfrak{A})^g}$$

So τ occurs in $\pi\,|_K$, and corresponds to the double coset $K\mathfrak{K}(\mathfrak{A})$. Restriction to K forgets tensoring with unramified quasicharacters, so if $\pi' \in \mathfrak{I}(\pi)$, then $\pi'\,|_K$ contains τ.

Suppose π' contains τ, then restriction to J will contain λ and by [5](6.2.3) $\pi' \cong \pi \otimes \chi \circ \det_A$, for some unramified quasicharacter χ of $F^\times$.

If τ was contained in π more than once, then by restricting to J, we would get that λ was contained in π more than once.

$$\pi\,|_J \cong \bigoplus_{h \in J\backslash G/E^\times J} \operatorname{Ind}_{J \cap J^h}^J \lambda^h\,|_{J \cap J^h}$$

Since $I_G(\lambda, \lambda|J) = E^\times J$, λ will only be a summand of the representation coming from the double coset $J.1.E^\times J$, which is isomorphic to λ. So λ occurs with multiplicity one, and hence τ occurs with multiplicity one. $\square$

4 Key

Suppose τ is any irreducible representation of K occurring in π. Since

$$\pi\,|_K \cong \bigoplus_{g \in K\backslash G/\mathfrak{K}(\mathfrak{A})} \operatorname{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^K \rho^g\,|_{K \cap \mathbf{U}(\mathfrak{A})^g}$$

then for some $g \in Kg\mathfrak{K}(\mathfrak{A})$ we have $\langle \tau, \operatorname{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^K \rho^g \rangle_K \neq 0$.

Let (J, λ) be a simple type, with the simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that $\rho \cong \operatorname{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$.

$$\rho\,|_{\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}}} \cong \bigoplus_{u \in \mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \backslash \mathbf{U}(\mathfrak{A})/J} \operatorname{Ind}_{J^u \cap K^{g^{-1}}}^{\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}}} \lambda^u$$

Hence

$$\operatorname{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^K \rho^g \cong \bigoplus_{u \in \mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \backslash \mathbf{U}(\mathfrak{A})/J} \operatorname{Ind}_{K \cap J^{ug}}^K \lambda^{ug}$$

We have the freedom to replace (J, λ), with (J^u, λ^u), for any $u \in \mathbf{U}(\mathfrak{A})$ and τ is a summand of at least one of the representations on the right. So we have the following proposition:

Proposition 4.1. *Suppose τ is an irreducible representation of K and g is a fixed representative of $Kg\mathfrak{K}(\mathfrak{A})$, such that $\langle \tau, \operatorname{Ind}_{K\cap \mathbf{U}(\mathfrak{A})^g}^{K} \rho^g \rangle_K \neq 0$, then there exists a simple type (J, λ), with the simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that*

$$\rho \cong \operatorname{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda \text{ and } \langle \tau, \operatorname{Ind}_{K\cap J^g}^{K} \lambda^g \rangle_K \neq 0$$

Moreover, suppose that for every irreducible summand ξ of $\lambda\,|_{J\cap K^{g^{-1}}}$ there exists a smooth irreducible representation λ' of J, such that

$$\langle \xi, \lambda' \rangle_{J\cap K^{g^{-1}}} \neq 0 \text{ and } I_G(\lambda', \lambda | J) = \emptyset$$

or a smooth irreducible representation θ' of $H^1 = H^1(\beta, \mathfrak{A})$, such that

$$\theta'\,|_{H^1\cap K^{g^{-1}}} = \theta\,|_{H^1\cap K^{g^{-1}}} \text{ and } I_G(\theta', \theta | H^1) = \emptyset$$

where $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and $\langle \theta, \lambda \rangle_{H^1} \neq 0$, then τ cannot be a type.

Proof. The first part of proposition is immediate from above. Suppose τ is a type. Let $\tilde{\tau}$ be an extension of τ to $F^\times K$, such that $\pi\,|_{F^\times K}$ contains $\tilde{\tau}$, then according to [4](5.2),

$$\text{c-Ind}_{F^\times K}^{G} \tilde{\tau} \cong \coprod \pi \otimes \chi_i \circ \det$$

where χ_i are finitely many unramified quasicharacters of $F^\times$. So $\pi\,|_J$ will contain all irreducible representations occurring in

$$\operatorname{Res}_J^G \text{c-Ind}_{F^\times K}^{G} \tilde{\tau} \cong \bigoplus_{h\in J\backslash G/F^\times K} \operatorname{Ind}_{J\cap K^h}^{J} \tau^h\,|_{J\cap K^h}$$

and $\pi\,|_{H^1}$ will contain all irreducible representations occurring in

$$\operatorname{Res}_{H^1}^G \text{c-Ind}_{F^\times K}^{G} \tilde{\tau} \cong \bigoplus_{h\in H^1\backslash G/F^\times K} \operatorname{Ind}_{H^1\cap K^h}^{H^1} \tau^h\,|_{H^1\cap K^h}$$

Now

$$\langle \tau, \lambda^g \rangle_{K\cap J^g} = \langle \tau, \operatorname{Ind}_{K\cap J^g}^{K} \lambda^g \rangle_K \neq 0$$

so

$$\langle \lambda, \tau^{g^{-1}} \rangle_{J\cap K^{g^{-1}}} \neq 0$$

Let ξ be an irreducible representation of $J \cap K^{g^{-1}}$, such that

$$\langle \lambda, \xi \rangle_{J\cap K^{g^{-1}}} \neq 0 \text{ and } \langle \xi, \tau^{g^{-1}} \rangle_{J\cap K^{g^{-1}}} \neq 0$$

By assumption there exists λ', such that

$$\langle \lambda', \xi \rangle_{J \cap K^{g^{-1}}} \neq 0 \text{ and } I_G(\lambda', \lambda | J) = \emptyset$$

So

$$\langle \lambda', \mathrm{Ind}_{J \cap K^{g^{-1}}}^{J} \tau^{g^{-1}} \rangle_J = \langle \lambda', \tau^{g^{-1}} \rangle_{J \cap K^{g^{-1}}} \neq 0$$

That implies λ' occurs in $\pi\,|_J$. On the other hand we know that for some extension Λ of λ to $E^\times J$, we have $\pi \cong \text{c-Ind}_{E^\times J}^{G} \Lambda$, so

$$\pi\,|_J \cong \bigoplus_{h \in J \backslash G / E^\times J} \mathrm{Ind}_{J \cap J^h}^{J} \lambda^h\,|_{J \cap J^h}$$

That implies that λ and λ' must intertwine in G, which is a contradiction.

We deal similarly with θ'. If $\lambda \cong \kappa \otimes \sigma$, then by unravelling all the definitions we have

$$\lambda\,|_{H^1} \cong (\dim \sigma \dim \kappa)\theta$$

Hence

$$\langle \theta, \tau^{g^{-1}} \rangle_{H^1 \cap K^{g^{-1}}} \neq 0$$

Then by the same argument as above, we show that $\theta'\,|_{H^1 \cap K^{g^{-1}}} = \theta\,|_{H^1 \cap K^{g^{-1}}}$ implies that θ' occurs in $\pi\,|_{H^1}$.

$$\pi\,|_{H^1} \cong \bigoplus_{h \in H^1 \backslash G / E^\times J} \mathrm{Ind}_{H^1 \cap J^h}^{H^1} \lambda^h\,|_{H^1 \cap J^h}$$

Hence $\langle \theta', \lambda^h \rangle_{H^1 \cap J^h} \neq 0$, for some $h \in G$, so $\langle \theta', \lambda^h \rangle_{H^1 \cap H^{1h}} \neq 0$, which implies $h \in I_G(\theta', \theta | H^1)$. And we obtain a contradiction. $\square$

Remark 4.2. *The conditions of the Proposition above might seem a little strange. So we give the following example. Suppose λ, λ' as above, and assume further, that (J, λ') is a simple type, with a simple stratum $[\mathfrak{A}, n', 0, \beta']$, occurring in a supercuspidal representation π'. Now the condition on irreducible summands translates into*

$$\langle \tau, \mathrm{Ind}_{K \cap J^g}^{K} \lambda'^g \rangle_K \neq 0 \text{ and } \langle \tau, \mathrm{Ind}_{K \cap J^g}^{K} \lambda^g \rangle_K \neq 0$$

Hence

$$\langle \tau, \pi' \rangle_K \neq 0 \text{ and } \langle \tau, \pi \rangle_K \neq 0$$

And since λ and λ' do not intertwine, we have

$$\pi' \ncong \pi \otimes \chi \circ \det$$

where χ is an unramified quasicharacter of $F^\times$. So τ cannot be a type. If $N = 2$, this situation arises in [9]§A.3.7 and §A.3.10.

The rest of the paper is concerned with picking a nice representative from a double coset $Kg\mathfrak{K}(\mathfrak{A})$, and constructing λ' and θ', when $Kg\mathfrak{K}(\mathfrak{A}) \neq K\mathfrak{K}(\mathfrak{A})$.

5 Representatives of $Kg\mathfrak{K}(\mathfrak{A})$

We will need to identify $\mathfrak{A}$ with the ring of block upper triangular matrices modulo $\mathfrak{p}_F$ in order to do explicit calculations. For that purpose we introduce the following notation.

Notation 5.1. *We will write $\mathbf{M}(m, \mathfrak{o}_F)$ for the ring of $m \times m$ matrices with coefficients in $\mathfrak{o}_F$. We will also write $\mathbf{M}(m, \mathfrak{p}_F^i) = \pi_F^i \mathbf{M}(m, \mathfrak{o}_F)$.*

Proposition 5.2. *Let $\mathfrak{A}$ be a principal hereditary $\mathfrak{o}_F$ order in A, and let $e = e(\mathfrak{A}|\mathfrak{o}_F)$. Suppose a double coset $Kg\mathfrak{K}(\mathfrak{A}) \neq K\mathfrak{K}(\mathfrak{A})$, then there exists a representative g of $Kg\mathfrak{K}(\mathfrak{A})$, such that one of the following holds:*

(A) The map $\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \to \mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is not surjective. Moreover, for some index j, $0 \leq j < e$, the image of $\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}}$ in $\mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})$, via the map:

$$\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \longrightarrow \mathbf{U}(\mathfrak{A}) \longrightarrow \mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})$$

is a proper parabolic subgroup of $\mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})$.

(B) The map $\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \to \mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is surjective, and

$$(h-1).L_{e-1} \subseteq L_{e+1}, \ \forall\, h \in \mathbf{U}^1(\mathfrak{A}) \cap K^{g^{-1}}$$

Proof. We identify $\mathfrak{A}$ with the ring of block upper triangular matrices modulo $\mathfrak{p}_F$, with respect to our basis $v_1, \ldots, v_N$ of V. Then K is identified with $\mathrm{GL}_N(\mathfrak{o}_F)$. Having made these identifications, we prove the lemma below:

Lemma 5.3. *There exists a representative $g \in Kg\mathfrak{K}(\mathfrak{A})$, such that g is a diagonal matrix and the diagonal entries $= (\pi_F^{\alpha_1}, \ldots, \pi_F^{\alpha_N})$, where*

$$\alpha_{i\frac{N}{e}+1} \geq \ldots \geq \alpha_{(i+1)\frac{N}{e}} \geq 0$$

for all $0 \leq i < e$ and one of the following holds:

(A) $\alpha_{j\frac{N}{e}+1} \neq \alpha_{(j+1)\frac{N}{e}}$, for some j, $0 \leq j < e$.

(B) $\alpha_{i\frac{N}{e}+1} = \alpha_{(i+1)\frac{N}{e}}$, for all i, $0 \leq i < e$, $\alpha_1 \geq 2$ and there exists an index j, such that $\alpha_k > 0$ if $k < j$ and $\alpha_k = 0$ if $k \geq j$, for all $1 \leq k \leq N$.

Proof. Let $\mathfrak{A}_m \subseteq \mathfrak{A}$ be the upper triangular matrices modulo $\mathfrak{p}_F$. Then the Iwahori decomposition tells us that G is a disjoint union of double cosets $\mathbf{U}(\mathfrak{A}_m)w\mathbf{U}(\mathfrak{A}_m)$, for $w \in \tilde{W} = W_0 \ltimes D$, where W_0 is the group of permutation matrices and D is is the group of diagonal matrices, whose eigenvalues are powers of π_F. We have $W_0 \leq K$ and using permutation matrices in K and $\mathbf{U}(\mathfrak{A})$, we can choose g to be diagonal with diagonal entries $= (\pi_F^{\alpha_1}, \ldots, \pi_F^{\alpha_N})$, where $\alpha_{i\frac{N}{e}+1} \geq \ldots \geq \alpha_{(i+1)\frac{N}{e}}$, for all $0 \leq i < e$. By multiplying g by an element of $F^\times$, we can ensure that $\alpha_i \geq 0$ for all $1 \leq i \leq N$ and at least one of them is equal to 0. If (A) is true, then we are done.

Otherwise, let $\Pi \in \mathfrak{K}(\mathfrak{A})$ be the element defined in Section 2.6, and let t be the following permutation matrix:

$$\begin{aligned} t : v_{\frac{N}{e}(e-1)+i} &\mapsto v_i, \text{ for } 1 \leq i \leq \frac{N}{e} \\ v_{j-\frac{N}{e}} &\mapsto v_j, \text{ for } \frac{N}{e} + 1 \leq j \leq N. \end{aligned}$$

We write $(\alpha_1, \ldots, \alpha_e)$ for the diagonal matrix $(\pi_F^{\alpha_1} I, \ldots, \pi_F^{\alpha_i} I, \ldots, \pi_F^{\alpha_e} I)$, and I is the $\frac{N}{e} \times \frac{N}{e}$ identity matrix. Let $\oplus$ be the map $\oplus : g \mapsto tg\Pi$, then

$$\oplus : (\alpha_1, \ldots, \alpha_e) \mapsto (\alpha_e + 1, \alpha_1, \ldots, \alpha_{e-1})$$

Similarly, let $\ominus$ be the map $\ominus : g \mapsto t^{-1}g\Pi^{-1}$, then

$$\ominus : (\alpha_1, \ldots, \alpha_e) \mapsto (\alpha_2, \ldots, \alpha_e, \alpha_1 - 1)$$

Let j be the smallest index such that $\alpha_j = 0$, then we replace g with $\oplus(g)$ $e - j$ times. If all $\alpha_i \leq 1$, then by replacing g with $\ominus(g)$ $e - 1$ times, we would obtain the identity matrix. The double coset $Kg\mathfrak{K}(\mathfrak{A}) \neq K\mathfrak{K}(\mathfrak{A})$, so let k be the smallest index such that $\alpha_k \geq 2$, then by replacing g with $\ominus(g)$ $k - 1$ times we get g of the required form. $\square$

If g satisfies part (A) of the Lemma, then g will also satisfy part (A) of the proposition. To see that it is enough to know what the $\frac{N}{e} \times \frac{N}{e}$ blocks on the diagonal of a matrix in $K \cap K^{g^{-1}}$ look like. Since g is chosen nicely, it is enough to do the computation for $e = 1$ and then reduce modulo $\mathfrak{p}_F$, which is easy.

If g satisfies part (B) of the Lemma, then g will also satisfy part (B) of the Proposition. Surjectivity follows by the same argument as in (A). For the second part we observe that every matrix $(A_{ij}) \in \mathbf{U}^1(\mathfrak{A}) \cap K^{g^{-1}}$, where $A_{ij} \in \mathbf{M}(\frac{N}{e}, \mathfrak{o}_F)$, has $A_{e1} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F^2)$. It is enough to do the calculation for

$K \cap K^{g^{-1}}$ and since g is chosen nicely, we may assume $e = N$, and then it is obvious. Since with respect to our fixed basis $v_1, \ldots, v_N$ of V:

$$L_{e-1} = \mathfrak{o}_F v_1 + \ldots + \mathfrak{o}_F v_{\frac{N}{e}} + \mathfrak{p}_F v_{\frac{N}{e}+1} + \ldots + \mathfrak{p}_F v_N$$

$$L_e = \mathfrak{p}_F v_1 + \ldots + \mathfrak{p}_F v_N$$

$$L_{e+1} = \mathfrak{p}_F v_1 + \ldots + \mathfrak{p}_F v_{\frac{N}{e}(e-1)} + \mathfrak{p}_F^2 v_{\frac{N}{e}(e-1)+1} + \ldots + \mathfrak{p}_F^2 v_N$$

we have

$$(h-1).L_{e-1} \subseteq L_{e+1},\ \forall\, h \in \mathbf{U}^1(\mathfrak{A}) \cap K^{g^{-1}}$$

$\square$

Definition 5.4. *Let $\mathfrak{A}$ be a principal hereditary $\mathfrak{o}_F$ order in A and g be a representative of a double coset $Kg\mathfrak{K}(\mathfrak{A})$. We say a pair $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has* ***property (A)*** *(resp.* ***property (B)****) if $Kg\mathfrak{K}(\mathfrak{A}) \neq K\mathfrak{K}(\mathfrak{A})$ and g satisfies 5.2 (A) (resp. 5.2 (B)).*

Remark 5.5. *If $e(\mathfrak{A}|\mathfrak{o}_F) = 1$, then all the double cosets have property (A). If $e(\mathfrak{A}|\mathfrak{o}_F) = N$, then all the double cosets have property (B). In particular, when $N = 2$, the two cases above correspond to the ones considered in [9].*

6 Double cosets with property (A)

If $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (A), then the map $\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}} \to \mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is not surjective. If $N = 2$, then $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (A) if and only if $e(\mathfrak{A}|\mathfrak{o}_F) = 1$, in which case $\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}}$ coincides with the group considered in [9]§A.3.7. For general N we show that if $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum, $E = F[\beta]$ and $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$, then the map

$$J(\beta, \mathfrak{A}) \cap K^{g^{-1}} \to J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$$

is not surjective. So we look for λ', satisfying the conditions of Proposition 4.1, of the form $\lambda' = \kappa \otimes \sigma'$, where σ' is a lift to J of an irreducible representation of $J/J^1 \cong \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. Since $J^1 \leq \operatorname{Ker} \sigma$, the restriction $\sigma\,|_{J \cap K^{g^{-1}}}$ depends only on the image of $J \cap K^{g^{-1}}$ in $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$.

Definition 6.1. *Suppose that $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (A), then we define $\mathcal{K}(g)$ to be the following subgroup of $\mathbf{U}(\mathfrak{A})$:*

$$\mathcal{K}(g) = (\mathbf{U}(\mathfrak{A}) \cap K^{g^{-1}})\mathbf{U}^1(\mathfrak{A}).$$

From the definition we get that $\mathcal{K}(g)$ is a parahoric subgroup of G, contained in $\mathbf{U}(\mathfrak{A})$. Since $\mathbf{U}^1(\mathfrak{A})$ is a subgroup of $\mathcal{K}(g)$, we have

$$\mathcal{K}(g) \cap J(\beta, \mathfrak{A}) = (\mathcal{K}(g) \cap \mathbf{U}(\mathfrak{B}_\beta))J^1(\beta, \mathfrak{A}).$$

For each i, $\mathbf{U}(\mathfrak{B}_\beta)$ acts on a $\mathfrak{k}_E$ vector space L_i/L_{i+1} and $\mathbf{U}(\mathfrak{A})$ acts on a $\mathfrak{k}_F$ vector space L_i/L_{i+1}. Let j be the index, such that the image of $\mathcal{K}(g)$ in $\mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})$ is a proper parabolic subgroup P. Let H be the image of $\mathcal{K}(g) \cap \mathbf{U}(\mathfrak{B}_\beta)$ in $\mathrm{Aut}_{\mathfrak{k}_E}(L_j/L_{j+1})$. We have the following diagram:

$$\begin{array}{ccccc}
\mathcal{K}(g) \cap \mathbf{U}(\mathfrak{B}_\beta) & \longrightarrow & \mathbf{U}(\mathfrak{B}_\beta) & \longrightarrow & \mathrm{Aut}_{\mathfrak{k}_E}(L_j/L_{j+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}(g) & \longrightarrow & \mathbf{U}(\mathfrak{A}) & \longrightarrow & \mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})
\end{array}$$

Horizontal arrows on the right are surjections, and all the other arrows are inclusions. Hence we get an injection $H \hookrightarrow P$. This injection will give us enough information about H, to handle $\sigma\,|_{\mathcal{K}(g)\cap J(\beta,\mathfrak{A})}$. We make the following definition.

Definition 6.2. *Let $\mathbb{F}_q$ be a finite field, H a subgroup $\mathrm{GL}_N(\mathbb{F}_q)$, $\mathbb{F}_{q^N}$ the unique extension of $\mathbb{F}_q$ of degree N, and*

$$S = \{h \in \mathrm{GL}_N(\mathbb{F}_q) : \chi_h(X) = f(X)^l, l \in \mathbb{N}, f(X) \textit{ irreducible over } \mathbb{F}_q\}$$

where $\chi_h(X)$ is a characteristic polynomial of h. Call H ***sufficiently small*** *if there exists a subfield $\mathbb{F}$ of $\mathbb{F}_{q^N}$, such that $[\mathbb{F}_{q^N} : \mathbb{F}] > 1$ and for all $h \in H \cap S$ the roots of $\chi_h(X)$ lie in $\mathbb{F}$.*

Lemma 6.3. *Let P be a proper parabolic subgroup of $\mathrm{GL}_N(\mathbb{F}_q)$, then P is sufficiently small.*

Proof. Without loss of generality we may assume that P is a maximal proper parabolic subgroup. As conjugation does not change the characteristic polynomial, we may further assume that P is a subgroup of block upper triangular matrices consisting of two blocks of size $a \times a$ and $b \times b$, where $a + b = N$. If $h \in P$, then the characteristic polynomial $\chi_h(X)$ of h can be written as a product $\chi_h(X) = f_1(X)f_2(X)$, where $\deg(f_1) = a$ and $\deg(f_2) = b$. Hence if $h \in P$ and $\chi_h(X) = f(X)^l$, where $f(X)$ is irreducible over $\mathbb{F}_q$, then $\deg(f)$ divides a and $\deg(f)$ divides b, so the roots of $f(X)$ lie in $\mathbb{F}_{q^c}$, where $c = \gcd(a, b)$. As c divides N and $c < N$, we deduce that P is sufficiently small. $\square$

Remark 6.4. *One might think that every sufficiently small subgroup is contained in a proper parabolic subgroup. The following example shows that this is not the case. Choose* a, $N > 1$, *such that* $\gcd(a, N) = 1$, *then* $\mathrm{GL}_N(\mathbb{F}_q)$ *is a sufficiently small subgroup of* $\mathrm{GL}_N(\mathbb{F}_{q^a})$, *with* $\mathbb{F} = \mathbb{F}_{q^N}$. *It cannot be contained in any proper parabolic subgroup of* $\mathrm{GL}_N(\mathbb{F}_{q^a})$, *since* $\mathbb{F}_{q^{Na}}$ *is the smallest extension of* $\mathbb{F}_{q^a}$ *containing* $\mathbb{F}_{q^N}$. *It is also not hard to construct an embedding* $\iota : \mathrm{GL}_N(\mathbb{F}_{q^a}) \hookrightarrow \mathrm{GL}_{Na}(\mathbb{F}_q)$, *such that* $\iota(\mathrm{GL}_N(\mathbb{F}_q))$ *is contained in a proper parabolic subgroup of* $GL_{Na}(\mathbb{F}_q)$, *which is the case considered in the Lemma below.*

Lemma 6.5. *Let* W *be an* $\mathbb{F}_{q^a}$ *vector space of dimension* N, *which we also consider as an* $\mathbb{F}_q$ *vector space. So we get an embedding of algebras*

$$\iota : \mathrm{End}_{\mathbb{F}_{q^a}}(W) \hookrightarrow \mathrm{End}_{\mathbb{F}_q}(W)$$

Let H *be a subgroup of* $\mathrm{Aut}_{\mathbb{F}_{q^a}}(W)$, *such that* $\iota(H)$ *is contained in a proper parabolic subgroup* P *of* $\mathrm{Aut}_{\mathbb{F}_q}(W)$, *then* H *is a sufficiently small subgroup of* $\mathrm{Aut}_{\mathbb{F}_{q^a}}(W)$.

Proof. Let $h \in H$ and suppose that the characteristic polynomial $\chi_h(X)$ of h in $\mathrm{End}_{\mathbb{F}_{q^a}}(W)$ is a power of a polynomial $f(X)$, which is irreducible over $\mathbb{F}_{q^a}$. Let $\mathbb{F}_{q^b}$ be the field generated by the coefficients of $f(X)$ over $\mathbb{F}_q$. Define $\tilde{f}(X)$ to be

$$\tilde{f}(X) = \prod_{\xi \in \mathrm{Gal}(\mathbb{F}_{q^b}/\mathbb{F}_q)} f(X)^{\xi}$$

where $\mathrm{Gal}(\mathbb{F}_{q^b}/\mathbb{F}_q)$ acts on the coefficients of $f(X)$. Then $\tilde{f}(X) \in \mathbb{F}_q[X]$ and $\tilde{f}(X)$ is irreducible over $\mathbb{F}_q$. We claim that the characteristic polynomial of $\iota(h)$ in $\mathrm{End}_{\mathbb{F}_q}(W)$ is a power of $\tilde{f}(X)$. To see that it is enough to show that the minimal polynomial of $\iota(h)$ divides some power of $\tilde{f}(X)$, but $\tilde{f}(\iota(h)) = \iota(\tilde{f}(h))$, and now the claim is obvious.

We apply Lemma 6.3 to P and $\mathrm{Aut}_{\mathbb{F}_q}(W)$ and hence we get a subfield $\mathbb{F}$ of $\mathbb{F}_{q^{aN}}$, such that $[\mathbb{F}_{q^{aN}} : \mathbb{F}] > 1$, and roots of $\tilde{f}(X)$ and hence roots of $f(X)$, lie in $\mathbb{F}$. The field $\mathbb{F}$ does not depend on the choice of h, so H is a sufficiently small subgroup of $\mathrm{Aut}_{\mathbb{F}_{q^a}}(W)$.

$\square$

Corollary 6.6. *Suppose* $(g, Kg\mathfrak{K}(\mathfrak{A}))$ *has property (A). Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum, such that* $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$ *and let* $E = F[\beta]$. *Let* H *be the image of* $\mathcal{K}(g) \cap \mathbf{U}(\mathfrak{B}_\beta)$ *in* $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. *Then* H *is a sufficiently small subgroup of* $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$ *and if* E *is a totally ramified extension of*

F, then H is contained in a proper parabolic subgroup of $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. Moreover $H \neq \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$.

Proof. As $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (A), we can find a lattice L_j in the lattice chain defining $\mathfrak{A}$, such that the image of $\mathcal{K}(g)$ in $\mathrm{Aut}_{\mathfrak{k}_F}(L_j/L_{j+1})$ is a proper parabolic subgroup P. As $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$, we get that $L_{j+1} = \pi_E L_j$ and hence $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta) \cong \mathrm{Aut}_{\mathfrak{k}_E}(L_j/L_{j+1})$, so Lemma 6.5 applied to L_j/L_{j+1}, implies that H is a sufficiently small subgroup of $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$.

If E is a totally ramified extension of F, then $\mathfrak{k}_E = \mathfrak{k}_F$ and $H \leq P$, so H is contained in a proper parabolic subgroup. Note that, in this case $\dim_{\mathfrak{k}_E}(L_j/L_{j+1}) = 1$ would imply $e(\mathfrak{A}|\mathfrak{o}_F) = N$, and hence $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (B).

We can pick a polynomial $f(X) \in \mathfrak{k}_E[X]$ of degree $\dim_{\mathfrak{k}_E}(L_j/L_{j+1})$, which is irreducible over $\mathfrak{k}_E$ and $f(X) \neq X$. From linear algebra we know that there exists some $h \in \mathrm{Aut}_{\mathfrak{k}_E}(L_j/L_{j+1})$, such that the characteristic polynomial of h equals to $f(X)$. That implies $H \neq \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$, as H is sufficiently small. □

Lemma 6.7. *For all integers $q > 1$ and $N > 1$ there exists a prime r such that r divides $q^N - 1$, but r does not divide $q^m - 1$, for all $0 < m < N$, except when $q = 2^i - 1$ and $N = 2$ or $q = 2$ and $N = 6$.*

Proof. This result is known as Zsigmondy's theorem. We refer the reader to [11]. □

Proposition 6.8. *Let σ be a cuspidal irreducible representation of $\mathrm{GL}_N(\mathbb{F}_q)$ affording a character $\mathcal{X}$, where $\mathbb{F}_q$ is a finite field with q elements and p is the characteristic of $\mathbb{F}_q$.*

Suppose H is a sufficiently small subgroup of $\mathrm{GL}_N(\mathbb{F}_q)$, and if $q = 2$ or $q = 3$, we further assume that H is contained in a proper parabolic subgroup of $\mathrm{GL}_N(\mathbb{F}_q)$. Then for every irreducible representation ξ of H, such that $\langle \xi, \sigma \rangle_H \neq 0$, there exists an irreducible representation σ' of $\mathrm{GL}_N(\mathbb{F}_q)$, such that $\sigma \ncong \sigma'$ and $\langle \xi, \sigma' \rangle_H \neq 0$.

Moreover, in all, except finitely many, cases we may choose σ' to be a cuspidal representation, such that $\sigma|_H \cong \sigma'|_H$.

Remark 6.9. *So H is small enough to not distinguish between two different cuspidal representations.*

Proof. We denote $\mathrm{GL}_N(\mathbb{F}_q)$ by Γ and let

$$S = \{h \in \Gamma : \chi_h(X) = f(X)^l, l \in \mathbb{N}, f(X) \text{ irreducible over } \mathbb{F}_q\}$$

where $\chi_h(X)$ is a characteristic polynomial of h. Let $\mathbb{F}_{q^N}$ be an extension of $\mathbb{F}_q$ of degree N. As H is a sufficiently small subgroup there exists a subfield $\mathbb{F}$ of $\mathbb{F}_{q^N}$, such that $[\mathbb{F}_{q^N} : \mathbb{F}] > 1$ and for every $h \in H \cap S$ the roots of the characteristic polynomial of h lie in $\mathbb{F}$. First of all we get rid of some easy cases:

If $N = 1$, then σ is a one dimensional representation. Let Ψ be a lift to $\mathbb{F}_q^\times$ of some non-trivial linear character of $\mathbb{F}_q^\times / \mathbb{F}^\times$, then $\sigma' = \sigma \otimes \Psi$, satisfies the conditions of the proposition.

So we may assume that $N \geq 2$. The proposition is false if and only if we can find an irreducible representation ξ of H such that $\mathrm{Ind}_H^\Gamma \xi \cong \sigma \oplus \ldots \oplus \sigma$. Suppose H is contained in some proper parabolic subgroup P. Let U be the unipotent radical of the parabolic subgroup opposite to P. Then $U \cap P = 1$ and hence $U \cap H = 1$, so $\langle \mathbb{1}, \mathrm{Ind}_H^\Gamma \xi \rangle_U \neq 0$, which implies that $\langle \mathbb{1}, \sigma \rangle_U \neq 0$, but σ is a cuspidal representation, so we obtain a contradiction.

In general we use character theory. The characters of the irreducible representations of Γ were first described in [8], but [10] is also very useful. The conjugacy classes of Γ are in one-to-one correspondence with isomorphism classes of $\mathbb{F}_q[X]$ modules W , such that $\dim_{\mathbb{F}_q} W = N$ and $X.w = 0$ implies that $w = 0$, see [10]§IV.2. Each $h \in \Gamma$ acts naturally on $\mathbb{F}_q^N$, and hence defines an $\mathbb{F}_q[X]$ module structure on $\mathbb{F}_q^N$, such that $X.w = hw$, for all $w \in \mathbb{F}_q^N$. We denote this module by W_h. Clearly, two elements $h_1, h_2 \in \Gamma$ are conjugate if and only if $W_{h_1} \cong W_{h_2}$. We may therefore write W_c instead of W_h, where c is the conjugacy class of h in Γ. In our case we are only interested in those conjugacy classes c, where the characteristic polynomial $\chi_h(X)$ of any $h \in c$ is a power of a polynomial $f(X)$, which is irreducible over $\mathbb{F}_q$. Since $\mathbb{F}_q[X]$ is a principal ideal domain, and $\chi_h(X)$ will anniolate W_c, we have

$$W_c \cong \bigoplus_{i=1}^{k} \mathbb{F}_q[X]/(f)^{\mu_i(c)}$$

That defines a partition $\mu(c) = (\mu_1(c), \mu_2(c), \ldots, \mu_k(c))$ of $\frac{N}{d}$, where d is the degree of $f(X)$.

We are now ready to describe the characters of cuspidal representations of Γ, as given in [6] and [7]. Let $\Psi : \mathbb{F}_{q^N}^\times \to \mathbb{C}^\times$, be an abelian character, such that $\Psi^{q^m - 1} \neq 1$ for all m dividing but not equal to N, then the following

class function $\mathcal{X}_\Psi$ is a character of a cuspidal representation of Γ.

$$\mathcal{X}_\Psi(h) = \begin{cases} 0 & : \quad h \notin S \\ (-1)^{k+N}\varphi_k(q^d)(\Psi(\alpha^q) + \ldots + \Psi(\alpha^{q^d})) & : \quad h \in S \end{cases}$$

where $\varphi_k(X) = (X-1)\ldots(X^{k-1}-1)$, $\varphi_1(X) = 1$ and if $\chi_h(X) = f(X)^l$, $f(X)$ irreducible over $\mathbb{F}_q$, then d is the degree of f, α is a root of f and k is the number of parts in the partition $\mu(c)$, given by the conjugacy class c of h. Conversely, any character $\mathcal{X}$ of a cuspidal representation of $\mathrm{GL}_N(\mathbb{F}_q)$ arises in this way. Moreover, if Θ is another abelian character $\Theta : \mathbb{F}_{q^N}^\times \to \mathbb{C}^\times$, such that $\Theta^{q^m-1} \neq 1$ for all m dividing but not equal to N, then $\mathcal{X}_\Psi = \mathcal{X}_\Theta$ if and only if $\Psi = \Theta^{q^m}$, for some $m \geq 0$.

Let Ψ be an abelian character, such that $\mathcal{X} = \mathcal{X}_\Psi$ and suppose there exists an abelian character $\Theta : \mathbb{F}_{q^N}^\times \to \mathbb{C}^\times$, such that $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$, $\Theta^{q^m-1} \neq 1$, for all m dividing but not equal to N, and $\Theta \neq \Psi^{q^m}$, for $m \geq 0$, then we take σ' to be a cuspidal representation of Γ corresponding to the character $\mathcal{X}_\Theta$. Since $\Theta \neq \Psi^{q^m}$, the representations σ and σ' are not isomorphic, and since $\mathcal{X}_\Psi(h) = \mathcal{X}_\Theta(h)$ for all $h \in H$, $\sigma|_H$ is isomorphic to $\sigma'|_H$.

In most cases, we can show such Θ exists, by counting characters with the desired properties. The argument below was shown to me by S. D. Cohen. Let $a = [\mathbb{F}_q : \mathbb{F}_p]$ and $b = [\mathbb{F} : \mathbb{F}_p]$. By Lemma 6.7, there will exist a prime r, such that r divides $p^{aN} - 1 = q^N - 1$, but r does not divide $p^m - 1$, for all $0 < m < N$, unless $aN = 2$ and $p = 2^i - 1$, or $aN = 6$ and $p = 2$.

Suppose we are not in one of these exceptional cases. If Θ is not an r-th power, then r divides the order of Θ, and hence $\Theta^{p^m-1} \neq 1$, for all $0 < m < aN$. In particular, $\Theta^{q^m-1} \neq 1$, for all m dividing, but not equal to N. Since $\mathbb{F}_{q^N}^\times$ is cyclic and r does not divide $|\mathbb{F}^\times|$, every abelian character of $\mathbb{F}^\times$ is a restriction of an abelian character of $\mathbb{F}_{q^N}^\times$, which is an r-th power. Hence there will be $(1-\frac{1}{r})\frac{q^N-1}{p^b-1}$ characters Θ, such that Θ is not an r-th power and $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$. In order to avoid $\Psi^{q^m} = \Theta$ for some $m \geq 0$, we need the following inequality to hold:

$$(1-\frac{1}{r})\frac{q^N-1}{p^b-1} > N$$

Since $\mathbb{F}$ is a proper subfield of $\mathbb{F}_{q^N}$ we have $|\mathbb{F}| \leq q^{\frac{N}{2}}$. So it is enough to prove

$$(1-\frac{1}{r})(q^{\frac{N}{2}}+1) > N$$

That can be done using induction on N, when $q \geq 4$, then $r \geq 2$, when $q = 3$ and $N \geq 2$, then $r \geq 5$. When $q = 2$, $N \geq 3$, then $r \geq 5$, since $3 = 2^2 - 1$.

We are left with the following cases: $\mathrm{GL}_2(\mathbb{F}_2)$, $\mathrm{GL}_6(\mathbb{F}_2)$, $\mathrm{GL}_3(\mathbb{F}_4)$, $\mathrm{GL}_2(\mathbb{F}_8)$ and $\mathrm{GL}_2(\mathbb{F}_q)$, where q is a prime and $q = 2^i - 1$.

If $q = 2$ or $q = 3$, then by our assumption on H, it is contained in a proper parabolic subgroup and we have already dealt with this.

If $N = 2$ and $q = 2^i - 1$, where q is a prime, $q > 3$, then $\frac{q+1}{2} > 3$. Therefore we may pick an abelian character Ξ of $\mathbb{F}_{q^2}^\times$, such that $\Xi\,|_{\mathbb{F}_q^\times} = 1$ and Ξ^2 is not one of the following characters: 1, Ψ^{q-1} or $\Psi^{2(q-1)}$. Let $\Theta = \Psi\Xi$, then $\Theta^{q-1} = 1$ implies $\Xi^2 = \Psi^{q-1}$, $\Theta = \Psi$ implies $\Xi^2 = 1$ and $\Theta = \Psi^q$ implies $\Xi^2 = \Psi^{2(q-1)}$.

If $N = 3$ and $q = 4$ or $N = 2$ and $q = 8$, let α be an element of order 63 in $\mathbb{F}_{64}^\times$ and let c be the conjugacy class in Γ corresponding $\mathbb{F}_q[X]$ module $W = \mathbb{F}_q[X]/(m_\alpha)$, where $m_\alpha(X)$ is the minimal polynomial of α over $\mathbb{F}_q$. If $\mathrm{Ind}_H^\Gamma \xi \cong \sigma \oplus \ldots \oplus \sigma$, for some representation ξ of H, then $\mathcal{X}(c) = 0$, since H does not meet c in Γ, as α is not contained in any proper subfield of $\mathbb{F}_{64}$.

If $N = 2$ and $q = 8$, that implies $\Psi(\alpha) + \Psi(\alpha^8) = 0$, so 2 divides the order of Ψ , which is impossible.

If $N = 3$ and $q = 4$, then $\Psi(\alpha) + \Psi(\alpha^4) + \Psi(\alpha^{16}) = 0$, which implies Ψ has order 9, since $X^5 + X + 1 = (X^3 - X^2 + 1)(X^2 + X + 1)$ and $\Psi(\alpha)$ is a root of unity in $\mathbb{C}$. If $\mathbb{F} = \mathbb{F}_8$ we may take $\Theta = \Psi^2$, since $\Theta\,|_{\mathbb{F}_8^\times} = \Psi\,|_{\mathbb{F}_8^\times} = 1$ and $2 \not\equiv 4^m \pmod 9$. If $\mathbb{F} = \mathbb{F}_4$, then we may choose $\Theta = \Psi\Xi$, where Ξ is any character of order 7. $\square$

Remark 6.10. *When π is a supercuspidal representation of $\mathrm{GL}_2(F)$, it is enough to prove the proposition above for $\mathrm{GL}_1(\mathbb{F}_q)$, see [9] §A.3.7.*

We recall the following definition.

Definition 6.11. *[5](8.1.2) A **split type of level** $(0,0)$ is a pair (K', ϑ), given as follows:*

(i) $\mathfrak{A}$ *is a hereditary* $\mathfrak{o}_F$ *order in* A

(ii) $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A}) = \mathcal{G}_1 \times \ldots \times \mathcal{G}_e$*, where* $\mathcal{G}_1 \cong \mathrm{GL}_{n_i}(\mathfrak{k}_F)$

(iii) $K' = \mathbf{U}(\mathfrak{A})$ *and* ϑ *is the inflation of an irreducible representation of* $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ *of the form* $\xi_1 \otimes \ldots \otimes \xi_e$*, where* ξ_i *is a cuspidal representation of* $\mathcal{G}_i$*, and* $\xi_i \not\cong \xi_j$*, for some* $i \neq j$*.*

Proposition 6.12. *Let π be a supercuspidal representation of G and $\mathfrak{A}$ a maximal hereditary $\mathfrak{o}_F$ order in A, i.e., $e(\mathfrak{A}|\mathfrak{o}_F) = 1$. Suppose the space of vectors fixed by $\mathbf{U}^1(\mathfrak{A})$, $\pi^{\mathbf{U}^1(\mathfrak{A})} \neq 0$, then $\pi^{\mathbf{U}^1(\mathfrak{A})}$ considered as a representation of $\mathbf{U}(\mathfrak{A})$ is a lift of an irreducible cuspidal representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$.*

Proof. If σ is a representation of $\mathbf{U}(\mathfrak{A})$, which is an irreducible summand of $\pi^{\mathbf{U}^1(\mathfrak{A})}$, then σ is a lift of an irreducible representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$. If σ is not a lift of a cuspidal representation, then there exists a hereditary $\mathfrak{o}_F$ order $\mathfrak{A}'$ in A, such that $\mathfrak{A}' \subset \mathfrak{A}$, the image of $\mathbf{U}(\mathfrak{A}')$ in $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is a proper parabolic subgroup and $\sigma\,|_{\mathbf{U}(\mathfrak{A}')}$ contains representation ξ described below:

$$\mathbf{U}(\mathfrak{A}')/\mathbf{U}^1(\mathfrak{A}') \cong \mathrm{GL}_{n_1}(\mathfrak{k}_F) \times \ldots \times \mathrm{GL}_{n_k}(\mathfrak{k}_F)$$

where $n_1 + \ldots + n_k = N$. Let ξ_i be cuspidal representations of $\mathrm{GL}_{n_i}(\mathfrak{k}_F)$ and let ξ be a lift of $\xi_1 \otimes \ldots \otimes \xi_k$ to $\mathbf{U}(\mathfrak{A}')$.

If $\xi_i \not\cong \xi_j$ for some i and j, then $(\mathbf{U}(\mathfrak{A}'), \xi)$ is a split type of level $(0,0)$ and by [5](8.4.1) we know that a supercuspidal representation cannot contain a split type. Hence $n_1 = \ldots = n_k$ and $\xi_1 \cong \ldots \cong \xi_k$. Then $(\mathbf{U}(\mathfrak{A}'), \xi)$ is a simple type, but it is not maximal, and by [5](6.2.1) we know that supercuspidal representations contain only maximal simple types.

So σ is a lift of a cuspidal representation. Hence $(\mathbf{U}(\mathfrak{A}), \sigma)$ is a maximal simple type occurring in π, so by [5](6.2.3) $\pi \cong \text{c-Ind}_{F^\times \mathbf{U}(\mathfrak{A})}^{G} \tilde{\sigma}$, where $\tilde{\sigma}$ is some extension of σ to $F^\times \mathbf{U}(\mathfrak{A})$. Hence, an irreducible representation σ' of $\mathbf{U}(\mathfrak{A})$ will occur in $\pi\,|_{\mathbf{U}(\mathfrak{A})}$ if and only if $I_G(\sigma', \sigma|\mathbf{U}(\mathfrak{A})) \neq \emptyset$. If σ' is another irreducible summand of $\pi^{\mathbf{U}^1(\mathfrak{A})}$, then from above we get that σ' is a lift of a cuspidal representation and σ' intertwines with σ in G. By [5](5.7.1) there exists an $x \in G$, such that $\mathbf{U}(\mathfrak{A}) = \mathbf{U}(\mathfrak{A})^x$ and $\sigma' \cong \sigma^x$, since $(\mathbf{U}(\mathfrak{A}), \sigma')$ is a simple type. As $\mathfrak{A}$ is a maximal $\mathfrak{o}_F$ order in A, we get that $x \in F^\times \mathbf{U}(\mathfrak{A})$, and hence $\sigma' \cong \sigma$.

Therefore all irreducible factors of $\pi^{\mathbf{U}^1(\mathfrak{A})}$ will be isomorphic to σ, but Proposition 3.1 implies that σ occurs in π with multiplicity one. So $\pi^{\mathbf{U}^1(\mathfrak{A})} \cong \sigma$. ☐

Corollary 6.13. *Let $\mathfrak{A}$ be a maximal hereditary $\mathfrak{o}_F$ order in A, σ a lift of an irreducible cuspidal representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ and let σ' be a lift of any irreducible representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$, then σ and σ' intertwine in G if and only if $\sigma' \cong \sigma$.*

Proof. Apply Proposition 6.12 to $\text{c-Ind}_{F^\times \mathbf{U}(\mathfrak{A})}^{G} \tilde{\sigma}$, where $\tilde{\sigma}$ is any extension of σ to $F^\times \mathbf{U}(\mathfrak{A})$. ☐

Corollary 6.14. *Let $\mathfrak{A}$ be a maximal hereditary $\mathfrak{o}_F$ order in A, σ a lift of an irreducible cuspidal representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ and let $\mathcal{K}$ be a compact open subgroup of $\mathbf{U}(\mathfrak{A})$, such that its image in $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is a sufficiently small subgroup of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$. Moreover, if $q_F = 2$ or $q_F = 3$, then we assume further that the image of $\mathcal{K}$ in $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is contained in a proper parabolic subgroup of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$. Then for every irreducible summand ξ*

of $\sigma\,|_{\mathcal{K}}$ there exists a lift σ' of an irreducible representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$, *such that* $\langle \xi, \sigma' \rangle_K \neq 0$ *and* $I_G(\sigma, \sigma' | \mathbf{U}(\mathfrak{A})) = \emptyset$.

Proof. This is immediate from Proposition 6.8 and Corollary 6.13. □

The following proposition can be easily obtained by making some cosmetic changes to [5](5.3.2).

Proposition 6.15. *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum,* $J = J(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$ *and* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. *Let* η *be the unique representation of* J^1 *containing* θ *and let* κ *be a* β*-extension of* η. *Let* ζ *and* ζ' *be two lifts to* J *of irreducible representations of* $J/J^1 \cong \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. *Suppose, that* $I_{B^\times}(\zeta, \zeta' | \mathbf{U}(\mathfrak{B}_\beta)) = \emptyset$, *then* $I_G(\kappa \otimes \zeta, \kappa \otimes \zeta' | J) = \emptyset$.

We return to ideas and notations of Section 4.

Proposition 6.16. *Suppose* $(g, Kg\mathfrak{K}(\mathfrak{A}))$ *has property (A) and let* τ *be an irreducible representation of* K, *such that* $\langle \tau, \operatorname{Ind}_{K \cap \mathbf{U}(\mathfrak{A})^g}^{K} \rho^g \rangle_K \neq 0$, *then* τ *cannot be a type.*

Proof. Let (J, λ) be a simple type, with the simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that $\rho \cong \operatorname{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$ and $\langle \tau, \operatorname{Ind}_{K \cap J^g}^{K} \lambda^g \rangle_K \neq 0$. Let $E = F[\beta]$. We have to consider two cases.

Suppose $e(\mathfrak{A}|\mathfrak{o}_F) = 1$ and $(J, \lambda) = (\mathbf{U}(\mathfrak{A}), \sigma)$, where σ is a lift of a cuspidal representation of $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$. Then $\mathcal{K}(g) \leq \mathbf{U}(\mathfrak{A})$ and the image of $\mathcal{K}(g)$ in $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is a proper parabolic subgroup, so by Corollary 6.14 and Proposition 4.1 τ cannot be a type.

Otherwise, $\lambda = \kappa \otimes \sigma$, where σ is a lift of a cuspidal representation of $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. Let H be the image of $\mathcal{K}(g) \cap J$ in $J/J^1 \cong \mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. By Corollary 6.6 H is a sufficiently small subgroup $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$. Moreover, if $q_E = 2$ or $q_E = 3$, then E is a totally ramified extension of F, so H is contained in a proper parabolic subgroup.

We will abuse the notation in the following way. Since $\mathbf{U}^1(\mathfrak{A})$ is a subgroup of $\mathcal{K}(g)$,

$$(J \cap \mathcal{K}(g))/J^1 \cong (\mathbf{U}(\mathfrak{B}_\beta) \cap \mathcal{K}(g))/\mathbf{U}^1(\mathfrak{B}_\beta)$$

we will not distinguish between representations of $J \cap \mathcal{K}(g)$ (resp. J) on which J^1 acts trivially and representations of $\mathbf{U}(\mathfrak{B}_\beta) \cap \mathcal{K}(g)$ (resp. $\mathbf{U}(\mathfrak{B}_\beta)$) on which $\mathbf{U}^1(\mathfrak{B}_\beta)$ acts trivially.

Let ξ be an irreducible summand of $\lambda\,|_{J \cap \mathcal{K}(g)}$, then $\langle \xi, \kappa \otimes \zeta \rangle_{J \cap \mathcal{K}(g)} \neq 0$, for some irreducible summand ζ of $\sigma\,|_{\mathbf{U}(\mathfrak{B}_\beta) \cap \mathcal{K}(g)}$. By Corollary 6.14 there exists a

lift σ' of an irreducible representation of $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$ to $\mathbf{U}(\mathfrak{B}_\beta)$ such that $\langle\zeta,\sigma'\rangle_{\mathbf{U}(\mathfrak{B}_\beta)\cap\mathcal{K}(g)} \neq 0$ and σ does not intertwine with σ' in $B_\beta^\times$. Let $\lambda' = \kappa\otimes\sigma'$, then $\langle\xi,\lambda'\rangle_{J\cap\mathcal{K}(g)} \neq 0$ and by Proposition 6.15 λ does not intertwine with λ' in G. The representation λ' is irreducible, since:

$$\dim\sigma' = \langle\kappa\otimes\sigma',\eta\rangle_{J^1} = \langle\kappa\otimes\sigma',\kappa\otimes\operatorname{Ind}_{J^1}^{J}\mathbb{1}\rangle_J = \sum_{\zeta'}\dim\zeta'\langle\kappa\otimes\sigma',\kappa\otimes\zeta'\rangle_J$$

where $\eta = \kappa\,|_{J^1}$ and the sum is taken over all the irreducible representations of $\mathbf{U}(\mathfrak{B}_\beta)/\mathbf{U}^1(\mathfrak{B}_\beta)$ lifted to J. The equality implies that $\langle\lambda',\lambda'\rangle_J = 1$. So by Proposition 4.1 τ cannot be a type. $\square$

7 Double cosets with property (B)

If $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (B) then the map $\mathbf{U}(\mathfrak{A})\cap K^{g^{-1}} \to \mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$ is surjective, so we have to do something different than in the previous section. If $N = 2$ the groups $\mathbf{U}^1(\mathfrak{A})\cap K^{g^{-1}}$ coincide with the ones considered in [9]§A.3.10. For general N we will show that if $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum, then $H^1(\beta,\mathfrak{A})\cap K^{g^{-1}} \neq H^1(\beta,\mathfrak{A})$ and will find θ' satisfying conditions of Proposition 4.1. Again it is more convenient to work with a larger subgroup than $\mathbf{U}^1(\mathfrak{A})\cap K^{g^{-1}}$.

Definition 7.1. *Let M be the following $\mathfrak{o}_F$-lattice in A:*

$$M = \{h\in\mathfrak{P} : hL_{e-1}\subseteq L_{e+1}\}$$

where $e = e(\mathfrak{A}|\mathfrak{o}_F)$. And let $\mathcal{K} = 1 + M$, be a subgroup of $\mathbf{U}^1(\mathfrak{A})$.

Since $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (B) $\mathbf{U}^1(\mathfrak{A})\cap K^{g^{-1}}$ is a subgroup of $\mathcal{K}$. Also, from the definition it is clear that $\mathbf{U}^2(\mathfrak{A})$ is a subgroup of $\mathcal{K}$.

Lemma 7.2. *Suppose $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum and $E = F[\beta]$, then*

$$\mathcal{K}\cap\mathbf{U}^1(\mathfrak{B}_\beta) = 1 + M_\beta,\ \textit{where}\ M_\beta = \{h\in\mathfrak{Q}_\beta : hL_{e_\beta-1}\subseteq L_{e_\beta+1}\}$$

and $e_\beta = e(\mathfrak{B}_\beta|\mathfrak{o}_E)$. Moreover, if $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$, then $\mathcal{K}\cap\mathbf{U}^1(\mathfrak{B}_\beta) = \mathbf{U}^2(\mathfrak{B}_\beta)$.

Proof. If $x\in\mathcal{K}\cap\mathbf{U}^1(\mathfrak{B}_\beta)$, then $x-1\in\mathfrak{Q}_\beta\cap M = \{h\in\mathfrak{Q}_\beta : hL_{e-1}\subseteq L_{e+1}\}$. Since $L_{i+me_\beta} = \pi_E^m L_i$, for all i, $e = e(E|F)e_\beta$ and x commutes with π_E, as $x\in B_\beta$, we have $x-1\in M_\beta$.

If $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$, then $L_i = \pi_E^i L_0$, for all i, and since x commutes with π_E, we have $(x-1)L_i\subseteq L_{i+2}$. That implies $x-1\in\mathfrak{Q}_\beta^2$, so $x\in\mathbf{U}^2(\mathfrak{B}_\beta)$. $\square$

We return to ideas and notations of Section 4.

Proposition 7.3. *Suppose $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (B) and let τ be an irreducible representation of K, such that $\langle \tau, \mathrm{Ind}_{K\cap \mathbf{U}(\mathfrak{A})^g}^{K} \rho^g\rangle_K \neq 0$. Moreover, let (J,λ) be a simple type, with the simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that $\rho \cong \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$ and $\langle \tau, \mathrm{Ind}_{K\cap J^g}^{K} \lambda^g\rangle_K \neq 0$. Suppose $r = -k_0(\beta, \mathfrak{A}) > 1$, then τ cannot be a type.*

Proof. Let $E = F[\beta]$. Since $r > 1$, [5](3.1.15) implies the following decompositions:

$$H^1(\beta,\mathfrak{A}) = \mathbf{U}^1(\mathfrak{B}_\beta)H^2(\beta,\mathfrak{A})$$
$$H^1(\beta,\mathfrak{A}) \cap \mathcal{K} = (\mathbf{U}^1(\mathfrak{B}_\beta) \cap \mathcal{K})H^2(\beta,\mathfrak{A})$$

as $\mathbf{U}^2(\mathfrak{A})$ is a subgroup of $\mathcal{K}$. Now (J,λ) is contained in a supercuspidal representation, so by [5](6.2.1) $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$. Hence by Lemma 7.2

$$\mathcal{K} \cap H^1(\beta,\mathfrak{A}) = H^2(\beta,\mathfrak{A})$$

Let θ be a simple character occurring in $\lambda\,|_{H^1}$. Since $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$ the map

$$H^1 \to H^1/H^2 \cong \mathbf{U}^1(\mathfrak{B}_\beta)/\mathbf{U}^2(\mathfrak{B}_\beta) \stackrel{\det_B}{\to} (1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^2)$$

is surjective. Let $\tilde{\mu}$ be any non-trivial abelian character

$$\tilde{\mu} : (1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^2) \to \mathbb{C}^\times$$

and μ be its lift to H^1. Let $\theta' = \theta\mu$, then $\theta'\,|_{H^1\cap\mathcal{K}} = \theta\,|_{H^1\cap\mathcal{K}}$.

$$I_G(\theta',\theta|H^1) \subseteq I_G(\theta',\theta|H^2) = I_G(\theta,\theta|H^2) = J^1B_\beta^\times J^1$$

by [5](3.3.2). By [5](3.2.5) θ' is a simple character, so again by [5](3.3.2)

$$I_G(\theta',\theta'|H^1) = J^1B_\beta^\times J^1, \text{ and } I_G(\theta,\theta|H^1) = J^1B_\beta^\times J^1$$

that implies that $J^1B_\beta^\times J^1 \subseteq I_G(\mu,\mu|H^1)$. Since all the representations above are 1-dimensional, we may write everything explicitly. From above, if θ' and θ intertwine in G, then there exists $x \in J^1B_\beta^\times J^1$, such that

$$\theta(h)\mu(h) = \theta(xhx^{-1}),\ \forall\, h \in H^1 \cap x^{-1}H^1x$$

Since, such x will intertwine θ with itself, we have

$$\mu(h) = 1,\ \forall\, h \in H^1 \cap x^{-1}H^1x$$

As H^1 is normal in J^1, we may assume $x = bj$, where $b \in B_\beta^\times$ and $j \in J^1$. Since $\mu(jhj^{-1}) = \mu(h)$, for all $h \in H^1$, the intertwining of θ' and θ would imply that

$$\mu(h) = 1,\ \forall\, h \in H^1 \cap b^{-1}H^1 b$$

By restricting to $\mathbf{U}^1(\mathfrak{B}_\beta)$, we get

$$\mathbf{U}^1(\mathfrak{B}_\beta) \cap \mathbf{U}^1(\mathfrak{B}_\beta)^b \leq \operatorname{Ker}\mu$$

Since μ extends to $\mathbf{U}(\mathfrak{B}_\beta)$ and $\mathbf{U}^1(\mathfrak{B}_\beta)$ is normal in $\mathbf{U}(\mathfrak{B}_\beta)$, we have

$$\mathbf{U}^1(\mathfrak{B}_\beta) \cap \mathbf{U}^1(\mathfrak{B}_\beta)^{b_1} \leq \operatorname{Ker}\mu,\ \forall\, b_1 \in \mathbf{U}(\mathfrak{B}_\beta) b \mathbf{U}(\mathfrak{B}_\beta)$$

We choose a basis, which identifies $\mathbf{U}(\mathfrak{B}_\beta)$ with $\mathrm{GL}_{\frac{N}{d}}(\mathfrak{o}_E)$, where $d = [E : F]$, and take b_1 to be a diagonal matrix with the eigenvalues equal to powers of π_E. Conjugation by b_1 will fix the group D of diagonal matrices in $\mathbf{U}^1(\mathfrak{B}_\beta)$ and $D \not\leq \operatorname{Ker}\mu$. Hence θ and θ' do not intertwine in G. By Proposition 4.1 τ cannot be a type. $\square$

Remark 7.4. *When $N = 2$ the arguments above are essentially [9]§A.3.9 and §A.3.10.*

If $k_0(\beta, \mathfrak{A}) = -1$, then $H^1(\beta, \mathfrak{A}) \neq \mathbf{U}^1(\mathfrak{B}_\beta)H^2(\beta, \mathfrak{A})$, and if $N = 2$ Henniart uses a result of Casselman, which is not available for $N > 2$, see [9]§A.3.11. So we need a new idea. We recall some definitions.

Definition 7.5. *[5](2.3.1) A stratum of the form $[\mathfrak{A}, n, n-1, b]$ is called **fundamental** if $b + \mathfrak{P}^{1-n}$ does not contain a nilpotent element of A.*

Let $[\mathfrak{A}, n, n-1, b]$ be a fundamental stratum. We choose a prime element π_F of F and set

$$y_b = b^{\frac{e}{m}} \pi_F^{\frac{n}{m}} + \mathfrak{P},$$

where $e = e(\mathfrak{A}|\mathfrak{o}_F)$, $m = \gcd(n, e)$. As an element of $\mathfrak{A}/\mathfrak{P}$, this depends only on the equivalence class of the stratum $[\mathfrak{A}, n, n-1, b]$. Let $\phi_b(X) \in \mathfrak{k}_F[X]$ be the characteristic polynomial of y_b considered as an element of $\operatorname{End}_{\mathfrak{k}_F}(L_0/L_e)$ via the canonical embedding $\mathfrak{A}/\mathfrak{P} \subset \operatorname{End}_{\mathfrak{k}_F}(L_0/L_e)$.

Definition 7.6. *[5](2.3.3) A fundamental stratum $[\mathfrak{A}, n, n-1, b]$ is called **split fundamental** if $\phi_b(X)$ has at least two distinct irreducible factors in $\mathfrak{k}_F[X]$. Otherwise, we say that $[\mathfrak{A}, n, n-1, b]$ is **non-split fundamental**.*

We start with the simplest case, when the simple stratum occurring in π is $[\mathfrak{A}, 1, 0, \beta]$. The following Lemmas are preparation for Proposition 7.11.

Lemma 7.7. *Suppose $\mathfrak{A}$ is a principal hereditary $\mathfrak{o}_F$ order in A, $e = e(\mathfrak{A}|\mathfrak{o}_F)$ and $b \in \mathfrak{P}^{-1}$. We identify $\mathfrak{A}$ with block upper triangular matrices modulo $\mathfrak{p}_F$ and write $b = (B_{ij})$, where $B_{ij} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F^{-1})$, for $1 \leq i, j \leq e$. Suppose $\pi_F B_{1e}, B_{21}, \ldots, B_{(e-1)e} \in \mathrm{GL}_{\frac{N}{e}}(\mathfrak{o}_F)$, then $[\mathfrak{A}, 1, 0, b]$ is a fundamental stratum. Moreover,*

$$\phi_b(X) = (\det(X - \pi_F B_{e(e-1)} \ldots B_{21} B_{1e}))^e \pmod{\mathfrak{p}_F}.$$

Proof. Both statements above depend only on the coset $b + \mathfrak{A}$. We also know that $b \in \mathfrak{P}^{-1}$, so we may assume that $B_{1e} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F^{-1})$, $B_{(i+1)i} \in \mathbf{M}(\frac{N}{e}, \mathfrak{o}_F)$, for $1 \leq i < e$, and $B_{ij} = 0$, otherwise.

Let Π be the element defined in the Section 2.6. Then Πb is a block diagonal matrix with the i-th block equal to $B_{(i+1)i}$ for $1 \leq i < e$ and the e-th block equal to $\pi_F B_{1e}$. By our assumption, the blocks on diagonal are in $\mathrm{GL}_{\frac{N}{e}}(\mathfrak{o}_F)$. So $\Pi b \in \mathbf{U}(\mathfrak{A})$, hence

$$b + \mathfrak{A} = \Pi^{-1}(\Pi b + \mathfrak{P}) \subset \Pi^{-1}\mathbf{U}(\mathfrak{A})$$

as $\nu_{\mathfrak{A}}(\Pi) = 1$. So $\nu_{\mathfrak{A}}(b) = -1$ and every element in $b + \mathfrak{A}$ is invertible, hence $[\mathfrak{A}, 1, 0, b]$ is a fundamental stratum.

Using block multiplication, we can calculate $\pi_F b^e$. Let $\pi_F b^e = (\tilde{B}_{ij})$, where $\tilde{B}_{ij} \in \mathbf{M}(\frac{N}{e}, \mathfrak{o}_F)$ for $1 \leq i, j \leq e$, then $\tilde{B}_{ij} = 0$, if $i \neq j$ and :

$$\begin{aligned}
\tilde{B}_{11} &= \pi_F B_{1e} B_{e(e-1)} B_{(e-1)(e-2)} \ldots B_{21}, \\
\tilde{B}_{22} &= \pi_F B_{21} B_{1e} B_{e(e-1)} B_{(e-1)(e-2)} \ldots B_{32}, \\
&\vdots \\
\tilde{B}_{ee} &= \pi_F B_{e(e-1)} B_{(e-1)(e-2)} \ldots B_{21} B_{1e}.
\end{aligned}$$

Hence

$$\phi_b(X) = (\det(X - \pi_F B_{e(e-1)} \ldots B_{21} B_{1e}))^e \pmod{\mathfrak{p}_F}.$$

□

Lemma 7.8. *Let $\mathfrak{A}$ be a principal hereditary $\mathfrak{o}_F$ order in A, $q_F = 2$ and $e(\mathfrak{A}|\mathfrak{o}_F) = \frac{N}{2}$. Suppose the stratum $[\mathfrak{A}, 1, 0, b]$ is fundamental and $\phi_b(X)$ is a power of $X^2 + X + 1$, then $[\mathfrak{A}, 1, 0, b]$ is equivalent to a simple stratum.*

Proof. The polynomial $X^2 + X + 1$ is irreducible over $\mathbb{F}_2$, so the stratum $[\mathfrak{A}, 1, 0, b]$ is non-split fundamental. From [5](2.3.4), we know that there exists a simple stratum $[\mathfrak{A}', n', n' - 1, \alpha]$ such that $b + \mathfrak{A} \subseteq \alpha + \mathfrak{P}'^{1-n'}$, moreover $\frac{2}{N} = \frac{n'}{e(\mathfrak{A}')}$, and the lattice chain defining $\mathfrak{A}'$ contains that defining $\mathfrak{A}$.

If $e(\mathfrak{A}'|\mathfrak{o}_F) = \frac{N}{2}$, then $\mathfrak{A} = \mathfrak{A}'$ and $n' = 1$. So $b + \mathfrak{A} = \alpha + \mathfrak{A}$ and we are done. Otherwise, $e(\mathfrak{A}'|\mathfrak{o}_F) = N$, $n' = 2$ and $b + \mathfrak{P}'^{-1} = \alpha + \mathfrak{P}'^{-1}$. Hence $\nu_{\mathfrak{A}'}(b) = -2$, so $\pi_F b^{\frac{N}{2}} \in \mathfrak{A}'$. Therefore the characteristic polynomial of $\pi_F b^{\frac{N}{2}}$ modulo $\mathfrak{p}_F$ is a power of $X-1$, but $\phi_b(X)$ associated to $[\mathfrak{A}, 1, 0, b]$ is also the characteristic polynomial of $\pi_F b^{\frac{N}{2}}$ modulo $\mathfrak{p}_F$ and it is a power of $X^2 + X + 1$. We get a contradiction. □

Lemma 7.9. *Suppose $[\mathfrak{A}, 1, 0, \beta]$ and $[\mathfrak{A}, 1, 0, \gamma]$ are simple strata, such that $I_G(\psi_\beta, \psi_\gamma|\mathbf{U}^1(\mathfrak{A})) \neq \emptyset$, then $\phi_\beta(X) = \phi_\gamma(X)$.*

Proof. In this case we have $H^1(\beta) = H^1(\gamma) = \mathbf{U}^1(\mathfrak{A})$, $\psi_\beta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and $\psi_\gamma \in \mathcal{C}(\mathfrak{A}, 0, \gamma)$. As ψ_β and ψ_γ intertwine in G we use [5](3.5.11) to get $x \in \mathbf{U}(\mathfrak{A})$ such that $\psi_\beta = \psi_\gamma^x$. That implies $\beta + \mathfrak{A} = x^{-1}\gamma x + \mathfrak{A}$, and as conjugation does not change characteristic polynomials we get the result. □

Definition 7.10. *For an $\mathfrak{o}_F$ lattice L in A, we define*

$$L^* = \{x \in A : \psi_A(xh) = 1 \textit{ for all } h \in L\}$$

Proposition 7.11. *Let $[\mathfrak{A}, 1, 0, \beta]$ be a simple stratum, $e = e(\mathfrak{A}|\mathfrak{o}_F) > 1$, $E = F[\beta]$, $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$ and M as in Definition 7.1. Then there exists $b \in M^*$ such that one of the following holds:*

1. *If $q_F > 2$ and $e < N$ or $q_F = 2$ and $e < \frac{N}{2}$ then the stratum $[\mathfrak{A}, 1, 0, \beta+b]$ is split fundamental.*
2. *If $q_F = 2$, $e = \frac{N}{2}$ and E is totally ramified over F then $[\mathfrak{A}, 1, 0, \beta + b]$ is equivalent to a simple stratum and $I_G(\psi_\beta, \psi_{\beta+b}|\mathbf{U}^1(\mathfrak{A})) = \emptyset$.*
3. *If $e = N$ or $q_F = 2$, $e = \frac{N}{2}$ and E is not totally ramified over F then $[\mathfrak{A}, 1, 0, \beta + b]$ is not fundamental and $I_G(\psi_\beta, \psi_{\beta+b}|\mathbf{U}^1(\mathfrak{A})) = \emptyset$.*

Remark 7.12. *Note, that $b \in M^*$ if and only if $\psi_\beta|_{\mathcal{K}} = \psi_{\beta+b}|_{\mathcal{K}}$.*

Proof. We identify $\mathfrak{A}$ with block upper triangular matrices modulo $\mathfrak{p}_F$. Since $\beta \in \mathfrak{P}^{-1}$, we can write β with respect to our fixed basis as a matrix (A_{ij}), $1 \leq i, j \leq e$, where $A_{1e} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F^{-1})$, $A_{ij} \in \mathbf{M}(\frac{N}{e}, \mathfrak{o}_F)$ for all $i \leq j+1$ and $(i,j) \neq (1,e)$ and $A_{ij} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F)$ otherwise. Let y be the matrix $(\tilde{A}_{ij})$, where $\tilde{A}_{(i+1)i} = A_{(i+1)i}$, for $1 \leq i < e$, $\tilde{A}_{1e} = A_{1e}$, and $\tilde{A}_{ij} = 0$, otherwise. So $y + \mathfrak{A} = \beta + \mathfrak{A}$. Let Π be the element defined in the Section 2.6. Since $\nu_{\mathfrak{A}}(\Pi) = 1$ we have

$$\Pi y \in \Pi(\beta + \mathfrak{A}) = \Pi\beta\mathbf{U}^1(\mathfrak{A}) \subset \mathbf{U}(\mathfrak{A}).$$

The matrix Πy is block diagonal with the i-th block equal to $A_{(i+1)i}$ for $1 \le i < e$ and the e-th block equal to $\pi_F A_{1e}$. So we can apply Lemma 7.7 to get that stratum $[\mathfrak{A}, 1, 0, \beta]$ is fundamental and

$$\phi_\beta(X) = \phi_y(X) = (\det(X - \pi_F A_{e(e-1)} \dots A_{21} A_{1e}))^e \pmod{\mathfrak{p}_F}.$$

Let $b = (B_{ij})$, $1 \le i, j \le e$, where $B_{1e} \in \mathbf{M}(\frac{N}{e}, \mathfrak{p}_F^{-1})$ and $B_{ij} = 0$ otherwise. Then from multiplication of blocks, we can see that $b \in M^*$. In each case we will find a matrix B_{1e}, such that conditions of proposition are satisfied. We note that, $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$ implies that the ramification index of E/F $e(E|F) = e(\mathfrak{A}|\mathfrak{o}_F)$.

1. If $q_F > 2$ and $e < N$ or $q_F = 2$ and $e < \frac{N}{2}$, then we can find a matrix $C \in \mathrm{GL}_{\frac{N}{e}}(\mathfrak{o}_F)$, such that the characteristic polynomial of C in modulo $\mathfrak{p}_F$ contains two distinct irreducible factors over $\mathfrak{k}_F$. Note that, this is not possible if $q_F = 2$ and $e = \frac{N}{2}$. Let

$$B_{1e} = \pi_F^{-1} A_{21}^{-1} \dots A_{e(e-1)}^{-1} C - A_{1e}$$

then Lemma 7.7 implies that the stratum $[\mathfrak{A}, 1, 0, \beta + b]$ is fundamental and $\phi_{\beta+b}(X) = (\det(X - C))^e \pmod{\mathfrak{p}_F}$, so $[\mathfrak{A}, 1, 0, \beta + b]$ is split fundamental.

2. If $q_F = 2$, $e = \frac{N}{2}$ and E is totally ramified over F, let $C \in \mathbf{M}(2, \mathfrak{o}_F)$ be a matrix such that the characteristic polynomial of C modulo $\mathfrak{p}_F$ is $X^2 + X + 1$. Let

$$B_{1e} = \pi_F^{-1} A_{21}^{-1} \dots A_{e(e-1)}^{-1} C - A_{1e}$$

so $[\mathfrak{A}, 1, 0, \beta + b]$ is fundamental, $\phi_{\beta+b}(X) = (\det(X - C))^e \pmod{\mathfrak{p}_F}$, which is a power of $X^2 + X + 1$, so by Lemma 7.8 $[\mathfrak{A}, 1, 0, \beta + b]$ is equivalent to a simple stratum. As $[\mathfrak{A}, 1, 0, \beta]$ is simple, we have $\nu_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A}) = -1$ and by [5](1.4.15) $\pi_F \beta^{\frac{N}{2}} + \mathfrak{p}_E$ generates $\mathfrak{k}_E$ over $\mathfrak{k}_F$. As E is totally ramified over F, we get that $\phi_\beta(X)$ is a power of $X - 1$. Then Lemma 7.9 implies that ψ_β and $\psi_{\beta+b}$ do not intertwine in G.

3. If $e = N$ or $q_F = 2$, $e = \frac{N}{2}$ and E is not totally ramified over F, then $[E : F] = N$. Let $J = J(\beta, \mathfrak{A})$, then $J^1(\beta) = H^1(\beta) = \mathbf{U}^1(\mathfrak{A})$, $J/\mathbf{U}^1(\mathfrak{A}) \cong \mathfrak{k}_E^\times$ and $\psi_\beta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Let λ be a simple type, such that

$$\lambda|_{\mathbf{U}^1(\mathfrak{A})} \cong \psi_\beta$$

and Λ any extension of λ to $E^\times J$. Let

$$\pi' = \text{c-Ind}_{E^\times J}^G \Lambda$$

so π' is a supercuspidal representation. As $\mathbf{U}^1(\mathfrak{A})$ is the unique maximal pro-p subgroup of J, we have:

$$\pi'|_{\mathbf{U}^1(\mathfrak{A})} \cong \bigoplus_{g\in\mathbf{U}^1(\mathfrak{A})\backslash G/E^\times J} \mathrm{Ind}_{\mathbf{U}^1(\mathfrak{A})\cap\mathbf{U}^1(\mathfrak{A})^g}^{\mathbf{U}^1(\mathfrak{A})} \psi_\beta^g|_{\mathbf{U}^1(\mathfrak{A})\cap\mathbf{U}^1(\mathfrak{A})^g}$$

Let $B_{1e} = -A_{1e}$, then the stratum $[\mathfrak{A}, 1, 0, \beta+b]$ is not fundamental. Suppose $\psi_{\beta+b}$ and ψ_β intertwine in G, then from above we know that $\psi_{\beta+b}$ occurs in $\pi'|_{\mathbf{U}^1(\mathfrak{A})}$.

Let $\mathfrak{A}_M = \mathrm{End}_{\mathfrak{o}_F}(L_0)$, then $\mathfrak{A}_M$ is a maximal hereditary $\mathfrak{o}_F$ order in A, $K = \mathbf{U}(\mathfrak{A}_M)$ and since $\mathbf{U}^1(\mathfrak{A}_M) = I_N + \mathbf{M}(N, \mathfrak{p}_F)$, where I_N is the identity matrix, we have $\psi_{\beta+b}|_{\mathbf{U}^1(\mathfrak{A}_M)} = 1$.

So $\pi'^{\mathbf{U}^1(\mathfrak{A}_M)} \neq 0$, hence Proposition 6.12 implies that $\pi'|_{\mathbf{U}(\mathfrak{A}_M)}$ contains σ, which is a lift of a cuspidal representation of $\mathbf{U}(\mathfrak{A}_M)/\mathbf{U}^1(\mathfrak{A}_M)$.

Since $(\mathbf{U}(\mathfrak{A}_M), \sigma)$ is another simple type occurring in π', by [5](6.2.4) there exists $g \in G$, such that $\mathbf{U}(\mathfrak{A}_M) = J^g$ and $\sigma \cong \lambda^g$. But J has a unique maximal pro-p subgroup, and $\mathbf{U}(\mathfrak{A}_M)$ does not, so that cannot happen. We get a contradiction, so $\psi_{\beta+b}$ does not intertwine with ψ_β. □

We recall the following definition.

Definition 7.13. *[5](8.1.1) A **split type of level** (x, x), $x > 0$, is a pair (K', ϑ) given as follows:*

(i) $[\mathfrak{A}, n, n-1, b]$ *is a split fundamental stratum in* A

(ii) $n > 0$, $\gcd(n, e(\mathfrak{A})) = 1$, $x = n/e(\mathfrak{A})$

(iii) $K' = \mathbf{U}^n(\mathfrak{A})$, $\vartheta = \psi_b$.

Corollary 7.14. *Suppose $(g, Kg\mathfrak{K}(\mathfrak{A}))$ has property (B) and let τ be an irreducible representation of K, such that $\langle\tau, \mathrm{Ind}_{K\cap\mathbf{U}(\mathfrak{A})^g}^K \rho^g\rangle_K \neq 0$.*

Moreover, suppose (J, λ) is a simple type with the simple stratum $[\mathfrak{A}, 1, 0, \beta]$, such that $\rho \cong \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$ and $\langle\tau, \mathrm{Ind}_{K\cap J^g}^K \lambda^g\rangle_K \neq 0$, then τ cannot be a type.

Proof. Since $[\mathfrak{A}, 1, 0, \beta]$ is a simple stratum, we have $r = -k_0(\beta, \mathfrak{A}) = 1$ and since (J, λ) is a simple type occurring in a supercuspidal representation, we have $e(\mathfrak{B}_\beta|\mathfrak{o}_E) = 1$. Also from the definitions of $J^1(\beta, \mathfrak{A})$ and $H^1(\beta, \mathfrak{A})$ [5](3.1.7) and (3.1.8) we get

$$J^1(\beta, \mathfrak{A}) = H^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{A})$$

and from the definition of simple characters [5](3.2.1) we have that the only simple character is ψ_β. Apply Proposition 7.11 to this situation, then we get an abelian character $\psi_{\beta+b}$ of $\mathbf{U}^1(\mathfrak{A})$, such that

$$\psi_{\beta+b}\,|_{\mathcal{K}} = \psi_\beta\,|_{\mathcal{K}}\,.$$

From the proof of Proposition 4.1, one can see that if τ is a type, then $\pi\,|_{\mathbf{U}^1(\mathfrak{A})}$ must contain all irreducible summands of $\mathrm{Ind}_{\mathbf{U}^1(\mathfrak{A})\cap K^{g^{-1}}}^{\mathbf{U}^1(\mathfrak{A})}\,\tau^{g^{-1}}$, so $\psi_{\beta+b}$ must occur in $\pi\,|_{\mathbf{U}^1(\mathfrak{A})}$.

If $q_F > 2$ and $e(\mathfrak{A}|\mathfrak{o}_F) < N$ or $q_F = 2$ and $e(\mathfrak{A}|\mathfrak{o}_F) < \frac{N}{2}$, then $[\mathfrak{A}, 1, 0, \beta{+}b]$ is a split fundamental stratum, so $(\mathbf{U}^1(\mathfrak{A}), \psi_{\beta+b})$ is a split type of level $(1/e, 1/e)$. But by [5](8.4.1) a supercuspidal representation cannot contain a split type. So τ is not a type.

In all the other cases of Proposition 7.11 $I_G(\psi_\beta, \psi_{\beta+b}|\mathbf{U}^1(\mathfrak{A})) = \emptyset$. So we apply Proposition 4.1 with $\theta = \psi_\beta$ and $\theta' = \psi_{\beta+b}$, and hence τ cannot be a type. □

We recall the following definition.

Definition 7.15. *[5](8.1.3) A* ***split type of level*** (x, y)***,*** $x > y > 0$*, is a pair* (K', ϑ) *given as follows:*

- *(i)* $[\mathfrak{A}, n, m, \beta]$ *is a simple stratum in* A *with* $E = F[\beta]$*,* $B = \mathrm{End}_E(V)$*,* $\mathfrak{B} = \mathfrak{A} \cap B$*,* $e_\beta = e(\mathfrak{B}|\mathfrak{o}_E)$*,* $\gcd(m, e_\beta) = 1$*,* $x = n/e(\mathfrak{A})$*,* $y = m/e(\mathfrak{A})$
- *(ii)* $K' = H^m(\beta, \mathfrak{A})$
- *(iii)* $\vartheta = \theta\psi_c$*, for some* $\theta \in \mathcal{C}(\mathfrak{A}, m-1, \beta)$ *and some* $c \in \mathfrak{P}^{-m}$*, such that the stratum* $[\mathfrak{B}, m, m-1, s_\beta(c)]$ *is split fundamental, where* s_β *denotes a tame corestriction on* A *relative to* E/F*.*

Proposition 7.16. *Suppose* $(g, Kg\mathfrak{K}(\mathfrak{A}))$ *has property (B) and let* τ *be an irreducible representation of* K*, such that* $\langle\tau, \mathrm{Ind}_{K\cap\mathbf{U}(\mathfrak{A})^g}^K\,\rho^g\rangle_K \neq 0$*.*

Moreover, let (J, λ) *be a simple type with the simple stratum* $[\mathfrak{A}, n, 0, \beta]$*, such that* $\rho \cong \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})}\,\lambda$*,* $\langle\tau, \mathrm{Ind}_{K\cap J^g}^K\,\lambda^g\rangle_K \neq 0$*. Suppose that* $r = -k_0(\beta, \mathfrak{A}) = 1$ *and* $n > 1$*, then* τ *cannot be a type.*

Proof. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, such that θ occurs in $\lambda\,|_{H^1(\beta)}$. By [5](3.2.3) there exists a simple stratum $[\mathfrak{A}, n, 1, \gamma]$, such that $[\mathfrak{A}, n, 1, \beta] \sim [\mathfrak{A}, n, 1, \gamma]$, $H^1(\beta, \mathfrak{A}) = H^1(\gamma, \mathfrak{A})$ and

$$\theta = \theta_0\psi_c,$$

where $\theta_0 \in \mathcal{C}(\mathfrak{A}, 0, \gamma)$ and $c = \beta - \gamma$. Since $\beta + \mathfrak{P}^{-1} = \gamma + \mathfrak{P}^{-1}$, we have $\nu_{\mathfrak{A}}(c) \geq -1$. If $\nu_{\mathfrak{A}}(c) \geq 0$, we would have $\beta + \mathfrak{A} = \gamma + \mathfrak{A}$, so $[\mathfrak{A}, n, 0, \beta] \sim [\mathfrak{A}, n, 0, \gamma]$. Since $[\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}, n, 0, \gamma]$ are both simple [5](2.4.1)(ii)(a) would imply $k_0(\beta, \mathfrak{A}) = k_0(\gamma, \mathfrak{A})$, but since $[\mathfrak{A}, n, 1, \gamma]$ is a simple stratum, we have $k_0(\gamma, \mathfrak{A}) \leq -2$ and $k_0(\beta, \mathfrak{A}) = -1$, hence

$$\nu_{\mathfrak{A}}(c) = -1.$$

That implies ψ_c extends to an abelian character of $\mathbf{U}^1(\mathfrak{A})$ and $\psi_c|_{\mathbf{U}^2(\mathfrak{A})} = 1$. Since $k_0(\gamma, \mathfrak{A}) \leq -2$, we have

$$H^1(\gamma) = \mathbf{U}^1(\mathfrak{B}_\gamma) H^2(\gamma)$$

and $\mathbf{U}^2(\mathfrak{A})$ is a subgroup of $\mathcal{K}$, so

$$\mathcal{K} \cap H^1(\beta) = \mathcal{K} \cap H^1(\gamma) = (\mathbf{U}^1(\mathfrak{B}_\gamma) \cap \mathcal{K}) H^2(\gamma) = \mathcal{K}_\gamma H^2(\gamma)$$

where $\mathcal{K}_\gamma = 1 + M_\gamma = \mathbf{U}^1(\mathfrak{B}_\gamma) \cap \mathcal{K}$ as in Lemma 7.2. Let

$$e_\gamma = e(\mathfrak{B}_\gamma | \mathfrak{o}_{F[\gamma]})$$

If $e_\gamma = 1$, then by Lemma 7.2 $\mathcal{K} \cap H^1(\gamma) = H^2(\gamma)$. Let $\theta' = \theta_0$, then $\theta'|_{H^1(\gamma) \cap \mathcal{K}} = \theta|_{H^1(\gamma) \cap \mathcal{K}}$ and by [5](3.5.12) $I_G(\theta, \theta' | H^1(\beta, \mathfrak{A})) = \emptyset$. So by Proposition 4.1 τ cannot be a type.

If $e_\gamma > 1$, we fix a continuous character $\psi_{F[\gamma]}$ of the additive group $F[\gamma]$ with the conductor $\mathfrak{p}_{F[\gamma]}$ and let

$$\psi_{B_\gamma}(b') = \psi_{F[\gamma]}(\mathrm{tr}_{B_\gamma / F[\gamma]}(b')),\ \forall\, b' \in B_\gamma$$

Then there exists a tame corestriction s_γ on A relative to $F[\gamma]/F$, such that

$$\psi_A(ab') = \psi_{B_\gamma}(s_\gamma(a) b'),\ \forall\, a \in A,\ \forall\, b' \in B_\gamma$$

In particular, for every $c' \in \mathfrak{P}^{-1}$ we have

$$\psi_{c', A}(b') = \psi_{s_\gamma(c'), B_\gamma}(b'),\ \forall\, b' \in \mathbf{U}^1(\mathfrak{B}_\gamma)$$

By [5](2.4.1)(iii) there exists a simple stratum $[\mathfrak{B}_\gamma, 1, 0, \delta]$ in B_γ, such that

$$[\mathfrak{B}_\gamma, 1, 0, s_\gamma(c)] \sim [\mathfrak{B}_\gamma, 1, 0, \delta]$$

We want to apply the Proposition 7.11 to $[\mathfrak{B}_\gamma, 1, 0, \delta]$. Let $B_{\gamma,\delta}$ be the B_γ-centraliser of $F[\gamma, \delta]$ and $\mathfrak{B}_{\gamma,\delta} = \mathfrak{B}_\gamma \cap B_{\gamma,\delta}$.

We claim that $e(\mathfrak{B}_{\gamma,\delta}|\mathfrak{o}_{F[\gamma,\delta]}) = 1$. By [5](2.2.8) we have

$$e(F[\gamma,\delta]|F) = e(F[\beta]|F).$$

Since $e(\mathfrak{B}_\beta|\mathfrak{o}_{F[\beta]}) = 1$, we also have

$$e(\mathfrak{A}|\mathfrak{o}_F) = e(F[\beta]|F).$$

And

$$e(\mathfrak{A}|\mathfrak{o}_F) = e(\mathfrak{B}_\gamma|\mathfrak{o}_{F[\gamma]})e(F[\gamma]|F).$$

Hence

$$e(F[\gamma,\delta]|F[\gamma]) = e(\mathfrak{B}_\gamma|\mathfrak{o}_{F[\gamma]}),$$

which proves the claim. So we can apply Proposition 7.11 to $[\mathfrak{B}_\gamma, 1, 0, \delta]$. We get $d \in \mathfrak{Q}_\gamma^{-1}$, such that $\psi_{\delta+d}$ is an abelian character of $\mathbf{U}^1(\mathfrak{B}_\gamma)$ and

$$\psi_{\delta+d}\,|_{\mathcal{K}_\gamma} = \psi_\delta\,|_{\mathcal{K}_\gamma}.$$

By [5](1.4.7) $s_\gamma : \mathfrak{P}^{-1} \to \mathfrak{Q}_\gamma^{-1}$ is surjective. Choose $b \in \mathfrak{P}^{-1}$, such that $s_\gamma(b) = d$, and let $\theta' = \theta_0\psi_{c+b}$. If $h \in \mathcal{K} \cap H^1(\gamma)$, then $h = h_1h_2$, for some $h_1 \in \mathcal{K}_\gamma$, $h_2 \in H^2(\gamma)$, and

$$\psi_{c+b,A}(h) = \psi_{c+b,A}(h_1) = \psi_{s_\gamma(c+b),B_\gamma}(h_1) = \psi_{\delta+d,B_\gamma}(h_1)$$

$$\psi_{c,A}(h) = \psi_{c,A}(h_1) = \psi_{s_\gamma(c),B_\gamma}(h_1) = \psi_{\delta,B_\gamma}(h_1)$$

From above $\psi_{c+b}\,|_{\mathcal{K}\cap H^1(\gamma)} = \psi_c\,|_{\mathcal{K}\cap H^1(\gamma)}$ and hence $\theta'\,|_{\mathcal{K}\cap H^1(\gamma)} = \theta\,|_{\mathcal{K}\cap H^1(\gamma)}$. So if τ was a type, then by arguments in Proposition 4.1, we would have that θ' occurs in $\pi\,|_{H^1(\beta)}$.

Suppose $q_{F[\gamma]} > 2$ and $e_\gamma[F[\gamma] : F] < N$ or $q_{F[\gamma]} = 2$ and $2e_\gamma[F[\gamma] : F] < N$, then the stratum $[\mathfrak{B}_\gamma, 1, 0, s_\gamma(c+b)]$ is split fundamental, so $(H^1(\gamma, \mathfrak{A}), \theta')$ is a split type of level $(n/e, 1/e)$, and by [5](8.4.1), a supercuspidal representation cannot contain a split type. So τ cannot be a type.

In all the other cases of Proposition 7.11, ψ_δ and $\psi_{\delta+d}$ do not intertwine in $B_\gamma^\times$. We will show that this implies that θ and θ' do not intertwine in G.

$$I_G(\theta', \theta|H^1(\gamma)) \subseteq I_G(\theta', \theta|H^2(\gamma)) = I_G(\theta_0, \theta_0|H^2(\gamma)) = J^1(\gamma)B_\gamma^\times J^1(\gamma)$$

by [5](3.3.2). By the same theorem $I_G(\theta_0, \theta_0|H^1(\gamma)) = J^1(\gamma)B_\gamma^\times J^1(\gamma)$ and θ_0 is an abelian character, so if h intertwines θ and θ' in G, it must also intertwine ψ_c and ψ_{c+b}. Both characters extend to $\mathbf{U}^1(\mathfrak{A})$ and $H^1(\gamma)$ is normal in $J^1(\gamma)$, so if $h = j_1b'j_2$, where $j_1, j_2 \in J^1(\gamma)$ and $b' \in B_\gamma^\times$, then

b' must also intertwine ψ_c and ψ_{c+b} in G and hence b' must intertwine the restrictions of these characters to $\mathbf{U}^1(\mathfrak{B}_\gamma)$ in $B_\gamma^\times$. So

$$b' \in I_{B^\times}(\psi_c, \psi_{c+b}|\mathbf{U}^1(\mathfrak{B}_\gamma)) = I_{B^\times}(\psi_\delta, \psi_{\delta+d}|\mathbf{U}^1(\mathfrak{B}_\gamma)) = \emptyset$$

That implies $I_G(\theta', \theta|H^1(\beta, \mathfrak{A})) = \emptyset$. By Proposition 4.1 τ cannot be a type. $\square$

Remark 7.17. *If $N = 2$, the case above does not have to be considered. Since, we can always find a smooth quasicharacter χ of $F^\times$, such that the simple stratum $[\mathfrak{A}, n, 0, \beta]$ occurring in $\pi \otimes \chi \circ \det$ has β minimal over F,i.e., $\nu_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A})$. Then it is easy to see, that it is enough to prove the unicity of types for $\pi \otimes \chi \circ \det$. I was told by Bushnell, that this works if and only if N is prime.*

8 Inertial correspondence

We collect all the bits together.

Theorem 8.1. *(Main) Let $G = \mathrm{GL}_N(F)$ and π be a smooth irreducible supercuspidal representation of G, then there exists a unique (up to isomorphism) smooth irreducible representation τ of $K = \mathrm{GL}_N(\mathfrak{o}_F)$, such that for any infinite dimensional smooth irreducible representation π' of G:*

$$\pi'|_K \text{ contains } \tau \Leftrightarrow \pi' \cong \pi \otimes \chi \circ \det$$

where χ is some unramified quasicharacter of $F^\times$.

Moreover, if (J, λ) is a simple type in a sense of [5], with the simple stratum $[\mathfrak{A}, n, 0, \beta]$, such that $\mathbf{U}(\mathfrak{A}) \leq K$ and $\pi \cong \text{c-Ind}_{E^\times J}^G \Lambda$, where $E = F[\beta]$ and Λ is an extension of λ to $E^\times J$, then $\tau \cong \mathrm{Ind}_J^K \lambda$.

Further, τ occurs in $\pi|_K$ with multiplicity one.

Proof. Let τ be any irreducible representation of K occurring in $\pi|_K$.

$$\pi|_K \cong \bigoplus_{g \in K\backslash G/\mathfrak{K}(\mathfrak{A})} \mathrm{Ind}_{K\cap\mathbf{U}(\mathfrak{A})^g}^K \rho^g|_{K\cap\mathbf{U}(\mathfrak{A})^g}$$

where $\rho = \mathrm{Ind}_J^{\mathbf{U}(\mathfrak{A})} \lambda$. Hence $\langle \tau, \mathrm{Ind}_{K\cap\mathbf{U}(\mathfrak{A})^g}^K \rho^g\rangle_K \neq 0$, for some representative $g \in G$.

If the double coset $Kg\mathfrak{K}(\mathfrak{A}) = K\mathfrak{K}(\mathfrak{A})$, then Proposition 3.1 says that $\tau \cong \mathrm{Ind}_J^K \lambda$, is a type and occurs in $\pi|_K$ with multiplicity one.

If the double coset $Kg\mathfrak{K}(\mathfrak{A}) \neq K\mathfrak{K}(\mathfrak{A})$, then we combine Propositions 6.16, 7.3, 7.14 and 7.16, to get that τ cannot be a type. That establishes uniqueness. $\square$

It also allows us to define a kind of inertial local Langlands correspondence for supercuspidals.

Corollary 8.2. *Let W_F be the Weil group of F, I_F the inertia subgroup, φ be a smooth N-dimensional representation of I_F, such that it extends to a smooth* <u>*irreducible*</u> *Frobenius semisimple representation of W_F, then there exists a unique (up to isomorphism) smooth irreducible representation $\tau(\varphi)$ of $K = \mathrm{GL}_N(\mathfrak{o}_F)$, such that for any smooth irreducible infinite dimensional representation π' of $G = \mathrm{GL}_N(F)$, $\tau(\varphi)$ occurs in π' with multiplicity at most* 1 *and :*

$$\pi'\,|_K \;\; \textit{contains } \tau(\varphi) \Leftrightarrow \mathrm{WD}(\pi)\,|_{I_F} \cong \varphi$$

where $\mathrm{WD}(\pi)$ is a Weil-Deligne representation of W_F corresponding to π' via the local Langlands correspondence.

Proof. Let φ_1 be an irreducible smooth Frobenius semisimple representation of W_F, such that $\varphi_1\,|_{I_F} \cong \varphi$ and let $\mathcal{L}$ denote the Langlands correspondence going from the Galois side to the automorphic side. Local Langlands correspondence preserves tensoring with quasicharacters, and irreducible N-dimensional representations of W_F are mapped to supercuspidal representations of G. So $\mathcal{L}(\varphi_1)$ is supercuspidal and if $\pi' \in \mathfrak{I}(\mathcal{L}(\varphi_1))$, then $WD(\pi')\,|_{I_F} \cong \varphi$. Conversely, if $\varphi_2 \cong \varphi_1 \otimes \chi$, then $\mathcal{L}(\varphi_2) \in \mathfrak{I}(\mathcal{L}(\varphi_1))$. So it is enough to prove the following statement:

Let φ_2 be a smooth Frobenius semisimple representations of W_F, such that $\varphi_2\,|_{I_F} \cong \varphi$, then $\varphi_2 \cong \varphi_1 \otimes \chi$, where χ is some unramified quasicharacter of $F^\times$.

Then Theorem 8.1 applied to $\mathfrak{I}(\mathcal{L}(\varphi_1))$ provides us with the unique $\tau = \tau(\varphi)$.

By tensoring with some unramified quasicharacter, we may assume that the image of $\varphi_1(W_F)$ in $\mathrm{GL}_N(\mathbb{C})$ is finite. By tensoring each irreducible factor of φ_2 by an unramified quasicharacter, we may assume that the image of $\varphi_2(W_F)$ in $\mathrm{GL}_N(\mathbb{C})$ is also finite. We can view φ_1 and φ_2 as representations of a finite group $H = W_F/(\mathrm{Ker}\,\varphi_1 \cap \mathrm{Ker}\,\varphi_2)$, and let I be the image of inertia in H. Then

$$0 \neq \langle \varphi_2, \varphi \rangle_I = \langle \varphi_2, \mathrm{Ind}_I^H \varphi \rangle_H = \langle \varphi_2, \varphi_1 \otimes \mathrm{Ind}_I^H \mathbb{1} \rangle_H$$

Since I is normal in H and H/I is cyclic, we have $\langle \varphi_2, \varphi_1 \otimes \chi \rangle_H \neq 0$, for some χ an abelian character of H/I. Since φ_1 is irreducible and has the same dimension as φ_2, we get $\varphi_2 \cong \varphi_1 \otimes \chi$. $\square$

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